Weak Harnack inequality for unbounded solutions to degenerate double-phase equations

M. Savchenko[1], **I. Skrypnik**[1], **Ye. Yevgenieva**[1,2]

[1]Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Sloviansk, Ukraine

[2]Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany

shan_maria@ukr.net, ihor.skrypnik@gmail.com, yevgenieva@mpi-magdeburg.mpg.de

We study double-phase parabolic equations of the form

$$u_t - \operatorname{div}\left(|\nabla u|^{p-1} + a(x,t)|\nabla u|^{q-1}\right) = 0, \quad (x,t) \in \Omega_T, \quad 2 < p < q, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a domain, $T > 0$, and the exponents satisfy $2 < p < q$. Our focus is on unbounded super-solutions to this equation.

The coefficient function $a(x,t) \geqslant 0$, defined on Ω_T, is assumed to satisfy the following continuity condition:

(A) There exist constants $A > 0$ and $\alpha \in (0,1]$ such that, for every cylinder

$$Q_{r,r^2}(x_0,t_0) := B_r(x_0,t_0) \times (t_0, t_0 + r^2) \subset Q_{8r,(16r)^2}(x_0,t_0) \subset \Omega_T$$

the oscillation of a satisfies

$$\operatorname{osc}_{Q_{r,r^2}(x_0,t_0)} a(x,t) \leqslant A r^{\alpha}.$$

It is well known that for integrands with (p,q)-growth, the gap between the exponents p and q must not be too large. Otherwise, in the case $q > \frac{np}{n-p}$ with $p < n$, unbounded minimizers may exist. For parabolic equations, local boundedness of solutions has been established under the condition $q \leqslant p\frac{n+2}{n}$ (see, for example, [3]). This upper bound on q arises naturally from the parabolic embedding theory.

The intrinsic Harnack inequality for bounded solutions to singular parabolic equations with (p,q)-growth was proved in [4]. In [2], using DiBenedetto's intrinsic scaling method, we establish a weak Harnack inequality for non-negative, possibly unbounded, super-solutions to equation (1), under assumptions similar to those in [1].

Our analysis is restricted to the degenerate case $p > 2$, while the singular case $p < 2 < q$ is left for future work. The main result of [2] is the weak Harnack inequality for non-negative super-solutions to the degenerate equation (1), assuming additionally that $u \in L^s_{\mathrm{loc}}(\Omega_T)$ for some sufficiently large exponent s. The precise formulation is given in the following theorem.

Theorem 1. *Let u be a weak super-solution to equation (1), and let condition (A) be fulfilled. Assume additionally that $u \in L^s(\Omega_T)$ and*

$$s \geqslant p - 2 + \frac{(q-p)(n+p)}{\alpha + p - q}.$$

Then there exist positive constants C_1, C_2, $C_3 > 0$ depending only on n, p, q, A and $d := \Big(\iint\limits_{\Omega_T} u^s dx dt\Big)^{\frac{1}{s}}$ such that for almost all $(x_0,t_0) \in \Omega_T$, either

$$\mathcal{I} := \frac{1}{|B_\rho(x_0)|} \int\limits_{B_\rho(x_0)} u(x,t_0)\, dx \leqslant C_1 \left\{ \rho + \rho\, \psi^{-1}_{Q_{14\rho,(14\rho)^2}(x_0,t_0)}\left(\frac{\rho^2}{T-t_0}\right)\right\},$$

or

$$\mathcal{I} \leqslant C_1 \inf_{B_{4\rho}(x_0)} u(\cdot, t),$$

for all time levels

$$t_0 + C_2\theta \leqslant t \leqslant t_0 + C_3\theta, \quad \theta := \frac{\rho^2}{\psi_{Q_{14\rho,(14\rho)^2}(x_0,t_0)}\big(\frac{\mathcal{I}}{\rho}\big)},$$

provided that $Q_{16\rho,(16\rho)^2}(x_0, t_0) \subset \Omega_T$. *Here* $\psi_Q(v) := \dfrac{\varphi_Q^+(v)}{v^2} = v^{p-2} + a_Q^+ v^{q-2}$, $v > 0$, $a_Q^+ := \max\limits_Q a(x,t)$ *and* $\psi_Q^{-1}(\cdot)$ *is inverse function to* $\psi_Q(\cdot)$.

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