Limit theorems for generalized convex hulls

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Over the past decades, considerable attention has been paid to the problem of generalized convexity and the related properties emerging from random point samples. Several generalizations of convex hulls have been proposed, with some extending this concept more broadly and others refining it in specific contexts.

The $(K, \mathbb{H})$-hull represents one of the broadest generalizations, encompassing several other formulations as special cases. The $(K, \mathbb{H})$-hull, introduced in [1], is defined in the following manner. Let K be a closed convex subset of $\mathbb{R}^d$, distinct from the entire space. Let $\mathbb{H}$ be a nonempty subset of $\mathbb{R}^d \times \mathbb{GL}_d$, where $\mathbb{GL}_d$ represents the group of all invertible linear transformations in $\mathbb{R}^d$. The $(K, \mathbb{H})$-hull of a set $A \subseteq \mathbb{R}^d$ is defined in [1] as follows:

$$\operatorname{conv}_{(K,\mathbb{H})}(A) = \bigcap_{(x,g)\in\mathbb{H}:\, A\subseteq g(K+x)} g(K+x),$$

where $g(B) = \{gz : z \in B\}$ and $B + x = \{z + x : z \in B\}$, for $(g, x) \in \mathbb{H}$ and $B \subseteq \mathbb{R}^d$.

The result presented in this talk demonstrates that the scaled normalization of the $(K, \mathbb{H})$-hull of a random sample, distributed according to a probability measure μ on K with a power-like behavior near the boundary ∂K of K, converges in distribution to a random closed set which can be viewed as the zero cell of a certain Poisson hyperplane tessellation with explicit intensity measure.

We assume that $K \in \mathcal{K}^d_{(0)}$, where $\mathcal{K}^d_{(0)}$ is the family of compact convex sets that contain the origin in their interior. The support plane of a convex closed set K with outer normal vector $u \neq 0$ is the following set:

$$H(K, u) := \{x \in \mathbb{R}^d : \langle x, u\rangle = h(K, u)\},$$

where $h(K, \cdot)$ is the support function of K. For $v \in \partial K$, the normal cone $N(K, v)$ to K at v is defined by

$$N(K, v) := \{u \in \mathbb{R}^d \setminus \{0\} : v \in H(K, u)\} \cup \{0\}.$$

Let $Nor(K)$ denote the normal bundle of the closed convex K, that is, a subset of $\partial K \times S^{d-1}$, which is the family of $(x, N(K, x) \cap S^{d-1})$ for $x \in \partial K$.

The following assumptions on the measure μ on K is borrowed from [2]. Let μ be a finite measure, supported by K, and satisfying:

(M1) The measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and has density f.

(M2) There exists an $\alpha > -1$ such that, for almost all $(a, u) \in \widehat{Nor}(K)$,

$$\lim_{t\downarrow 0} \frac{f(a + tu)}{t^\alpha} = \hat{g}(a, u) = g(a) \in [0, +\infty),$$

where the function $\hat{g}$ is strictly positive on a subset of ∂K of positive measure which is bounded almost everywhere and $\widehat{Nor}(K) = \{(x, -u) : (x, u) \in Nor(K)\}$.

Let $\Xi_n = \{\xi_1, \dots, \xi_n\}$ be a set of n independent random points with the common distribution μ. Let M_d denote the space of all real-valued $d \times d$ matrices, and let $\exp : M_d \mapsto \mathbb{GL}_d$ denote the standard matrix exponent. The following set

$$\mathfrak{X}_n := \{(x, C) \in \mathbb{R}^d \times M_d : \Xi_n \subseteq \exp(C)(K + x)\},$$

which is the main object of investigation in the present talk, has been introduced and studied in [1] in the case when μ is the uniform distribution on K. The importance of this set and its connections to $\mathrm{conv}_{(K,\mathbb{H})}(\Xi_n)$ and also to the Lie algebra of $\mathbb{R}^d \times \mathbb{GL}_d$ are explained in [1].

Let $\mathcal{P}_K$ denote Poisson process on $(0, \infty) \times Nor(K)$, where density ν is defined as product of an absolutely continuous measure on $(0, \infty)$ with density t^α and measure ν^* on $Nor(K)$, defined as

$$\nu^*(K, W \cap Nor(K)) = \int_{Nor(K)} \mathbf{1}_{(a,u)\in W}\, g(a) C_{d-1}(K, d(a, u)),$$

where $C_{d-1}(K, \cdot)$ is the curvature measure of K, see Section 2 in [3].

Theorem 1. *Assume that $(M1)$ and $(M2)$ hold. Additionally, assume that $K \in \mathcal{K}^d_{(0)}$, and let $\mathfrak{F}$ be a closed convex set in $\mathbb{R}^d \times M_d$ which contains the origin. Let $\gamma = (\alpha + 1)^{-1}$ be a real parameter, with $\alpha > -1$. The sequence of random closed sets $((n^\gamma \mathfrak{X}_n) \cap \mathfrak{F})_{n\in\mathbb{N}}$ converges in distribution in the space of closed subsets of $\mathbb{R}^d \times M_d$ endowed with the Fell topology to a random closed convex set $\check{\mathfrak{Z}}_K \cap \mathfrak{F}$, where*

$$\mathfrak{Z}_K := \bigcap_{(t,\eta,u)\in\mathcal{P}_K} \Big\{(x, C) \in \mathbb{R}^d \times M_d : \langle C\eta + x, u\rangle \le t\Big\}. \tag{1}$$

Corollary 1. *Convergence in distribution of random closed sets in Hausdorff metric also implies convergence of intrinsic volumes, so*

$$V_j(n^\gamma \mathfrak{X}_n \cap \mathfrak{F}) \xrightarrow{\mathrm{d}} V_j(\check{\mathfrak{Z}}_K \cap \mathfrak{F}) \quad as \quad n \to \infty,\ j = 0, 1, \dots d,$$

where V_j denotes j-th intrinsic volume.

As the set $\mathfrak{Z}_K$ is an element of space $\mathbb{R}^d \times M_d$, the latter space can be turned into a Euclidean one, given the proper scalar product, see Section 5.1 in [1]. Then, the set $\mathfrak{Z}_K$ can be viewed as the zero cell of the Poisson hyperplane tessellation in space $\mathbb{R}^d \times M_d$.

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