# Properties of the Existence of Solutions in Differential Inclusions

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Differential inclusions play a fundamental role in mathematical analysis and control theory, particularly in modeling dynamic systems governed by set-valued mappings. In our research, we have established existence theorems in optimal control theory through differential inclusion techniques. A primary concern in this field is the existence of solutions, which is contingent on various structural and topological properties of the inclusions. Key factors affecting solution existence include continuity conditions, compactness, upper semicontinuity, and convexity of the multifunction defining the inclusion. Notably, results such as Filippov's existence theorem and measurable selection principles provide rigorous criteria for ensuring the existence of viable trajectories [2]. Beyond theoretical considerations, these properties hold significant implications in applied mathematics, optimization, and control processes, shaping stability and feasibility in practical implementations [1,3,4].

This presentation focuses on the existence theorem for an optimal solution in an optimal control problem within the calculus of variations. Specifically, the study investigates the foundational existence results in the context of optimal control problems characterized by differential inclusions and boundary conditions. By leveraging differential inclusion techniques, we establish rigorous criteria that ensure the existence of feasible solutions, contributing to the broader theoretical framework of optimal control.

Let $X$ and $Y$ be two normed spaces. A set-valued mapping $F : X \to Y$ is defined as a function that assigns to each $x \in X$ a subset $F(x) \subset Y$. Such a mapping $F$ is termed convex if its graph, denoted by $gphF$, forms a convex subset of $X \times Y$. $L_1([0,1],\mathbb{R}^n)$ denotes the Banach space of integrable functions $x : [0,1] \to \mathbb{R}^n$ with the norm $|x(\cdot)|_{L_1} = \int_0^1 |x(t)|dt$. $C([0,1],\mathbb{R}^n)$ refers to the Banach space of continuous functions $x : [0,1] \to \mathbb{R}^n$, with the norm $|x(\cdot)|_C = \max\{|x(t)| : t \in [0,1]\}$. A function $x : [0,1] \to \mathbb{R}^n$ is said to be absolutely continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any countable collection of disjoint subintervals $[t_k^1, t_k^2]$ of $[0,1]$ satisfying $\sum(t_k^2 - t_k^1) < \delta$ we have $\sum |x(t_k^2) - x(t_k^1)| < \varepsilon$. The space of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$ is denoted by $AC([0,1],\mathbb{R}^n)$ with the norm $|x(\cdot)|_{AC} = |x(0)| + \int_0^1 |\dot{x}(t)|dt$.

**Definition 1.** A set-valued mapping $F$ is said to be *upper semi-continuous* at a point $x_0 \in X$ if there exists a neighborhood $\Omega$ of $x_0$ such that $F(\Omega) \subset C$ for every open set $C$ containing $F(x_0)$.

**Definition 2.** A subset $X \subset C([0,1],\mathbb{R}^n)$ is said to be *equicontinuous* if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for every $x(\cdot) \in X$, the inequality $|x(t_2) - x(t_1)| < \varepsilon$ holds whenever $t_1, t_2 \in [0,1]$ and $|t_2 - t_1| < \delta$.

**Theorem 1.** ***(Arzela-Ascoli)*** *If a set $X \subset C([0,1],\mathbb{R}^n)$ is both bounded and equicontinuous, then it satisfies a significant compactness property: it contains a uniformly convergent sequence $x_i(\cdot) \in X$, $i = 1, 2, \ldots$. Specifically, there exists a function $x(\cdot) \in C([0,1],\mathbb{R}^n)$ such that $|x_i(\cdot) - x(\cdot)|_C \to 0$ as $i \to \infty$ where $|\cdot|_C$ denotes the supremum norm on $C([0,1],\mathbb{R}^n)$.*

This theorem establishes a crucial compactness property for bounded sets of absolutely continuous functions. Specifically, if $X \subset C([0,1], \mathbb{R}^n)$ is a bounded collection of absolutely continuous functions satisfying the uniform bound $\dot{x}(t) \leq b$ for all $x(\cdot) \in X$ and $t \in [0,1]$, then the set contains a uniformly convergent subsequence.

Consider the differential inclusion

$$\dot{x}(t) \in \ F(x(t)), \ t \in [0,1] \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is an upper semi-continuous set-valued mapping with closed convex values contained within a ball of radius $b > 0$.

**Theorem 2.** *For any initial condition $x_0 \in \mathbb{R}^n$, there exist solutions to the differential inclusion (1) with the initial condition $x(0) = x_0$.*

To establish the existence of solutions to the differential inclusion (1) subject to the boundary conditions

$$x(0) \in C_0 \ , \ \ x(1) \in C_1 \tag{2}$$

where $C_0, C_1 \subset \mathbb{R}^n$ are closed sets, it is essential to analyze the structural properties of the set-valued mapping $F$. Given that $F$ is upper semi-continuous with closed convex values contained in a bounded region, standard existence theorems for differential inclusions, such as Filippov's theorem or viability principles, may be applicable to ensure the existence of solutions that satisfy the prescribed boundary constraints.

The set $S_{[0,1]}(F, C_0)$ consists of all absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$ that satisfy the inclusion in (1) for almost every $t \in [0,1]$ while adhering to the initial condition $x(0) \in C_0$. This set contains the complete collection of feasible trajectories governed by the differential inclusion.

**Theorem 3.** ***(Compactness and Convergence of Solutions)*** *Let $C_0 \subset \mathbb{R}^n$ be a compact set. Suppose there exists a sequence of solutions $x_k(\cdot) \in S_{[0,1]}(F, C_0)$. Due to the compactness properties of $C_0$ and the structural conditions of the set-valued mapping $F$, there exists a uniformly convergent subsequence $x_{kp}(\cdot)$ such that $x_{kp}(\cdot) \to x(\cdot)$ uniformly, where $x(\cdot) \in S_{[0,1]}(F, C_0)$ is a solution to the differential inclusion.*

**Theorem 4.** *Given that $C_0$ is a compact set and there exists a trajectory $x(\cdot)$ satisfying the differential inclusion (1) along with the boundary constraints$x(0) \in C_0$ and $x(1) \in C_1$, the existence of an optimal trajectory is ensured.*

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