PARASTROPHY ORBIT OF A WIP–QUASIGROUP

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An algebra $(Q, \circ, \overset{l}{\circ}, \overset{r}{\circ})$ with identities

$$(x \circ y) \overset{l}{\circ} y = x, \quad (x \overset{l}{\circ} y) \circ y = x, \quad x \overset{r}{\circ} (x \circ y) = y, \quad x \circ (x \overset{r}{\circ} y) = y$$

is called a *quasigroup*; the operation $(\circ)$ is *main*, $(\overset{l}{\circ})$, $(\overset{r}{\circ})$ are called *left* and *right divisions* of $(\circ)$.

Each inverse of an invertible operation is also invertible. All such operations are called *parastrophes* of $(\circ)$ and they are defined by

$$x_{1\sigma} \overset{\sigma}{\circ} x_{2\sigma} = x_{3\sigma} :\Leftrightarrow x_1 \circ x_2 = x_3,$$

where $\sigma \in S_3 := \{\iota, s, l, r, sl, sr\}$, $l := (13)$, $r := (23)$, $s := (12)$. In particular, the left and right divisions of $(\circ)$ are its parastrophes. It is easy to verify that

$$^{\sigma}(\overset{\sigma}{\circ}) = (\overset{\sigma\tau}{\circ})$$

holds for all $\sigma, \tau \in S_3$.

Let P be an arbitrary proposition in a class of quasigroup $\mathfrak{A}$. A proposition ${}^{\sigma}P$ is said to be a σ-parastrophe of P, if it can be obtained from P by replacing the main operation with its σ^{-1}-parastrophe.

Let ${}^{\sigma}\mathfrak{A}$ denote the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$. A set of all pairwise parastrophic classes is called a *parastrophy orbit* of $\mathfrak{A}$:

$$Po(\mathfrak{A}) = \{{}^{\sigma}\mathfrak{A} \mid \sigma \in S_3\}.$$

A parastrophy orbit of varieties is uniquely defined by one of its varieties. Parastrophy orbits were studied by Alla Lutsenko and Fedir Sokhatsky [1], [2].

A quasigroup $(Q, \circ)$ is said to be a *WIP–quasigroup* with respect to the permutation α of Q if

$$x \circ \alpha(y \circ x) = \alpha(y)$$

for all $x, y \in Q$ [3].

Theorem 1. *The parastrophy orbit of a WIP–quasigroup comprises six varieties:*

$$Po(\mathfrak{A}) = \{\mathfrak{A},\ {}^{s}\mathfrak{A},\ {}^{l}\mathfrak{A},\ {}^{r}\mathfrak{A},\ {}^{sl}\mathfrak{A},\ {}^{sr}\mathfrak{A}\}.$$

Variety	*Identity*
$\mathfrak{A}$	$x \circ \alpha(y \circ x) = \alpha(y)$
${}^{s}\mathfrak{A}$	$\alpha(x \circ y) \circ x = \alpha(y)$
${}^{l}\mathfrak{A}$	$\alpha(x \circ y) \circ \alpha(x) = y$
${}^{r}\mathfrak{A}$	$(x \circ y) \circ \alpha(x) = \alpha(y)$
${}^{sl}\mathfrak{A}$	$\alpha(x) \circ (y \circ x) = \alpha(y)$
${}^{sr}\mathfrak{A}$	$\alpha(x) \circ \alpha(y \circ x) = y$

Example 1. The quasigroup $(Q, \circ)$ with the following Cayley table:

$\circ$	1	2	3	4	5	6	7
1	1	4	5	3	2	6	7
2	5	1	4	6	7	3	2
3	4	5	1	7	6	2	3
4	2	6	7	1	3	4	5
5	3	7	6	2	1	5	4
6	6	2	3	5	4	7	1
7	7	3	2	4	5	1	6

is a WIP–quasigroup with respect to the permutation $\alpha = (1)(2534)(67)$ (see [3, p. 205]). That is, in this quasigroup, the identity of the variety $\mathfrak{A}$ holds for all $x, y \in Q$.

Let us show that the identities of the varieties ${}^{s}\mathfrak{A}$, ${}^{l}\mathfrak{A}$, ${}^{r}\mathfrak{A}$, ${}^{sl}\mathfrak{A}$, ${}^{sr}\mathfrak{A}$ do not hold for at least for one pair of $x, y \in Q$.

Put $x = 2$, $y = 3$:

$$
\begin{array}{ll}
{}^{s}\mathfrak{A} & \alpha(2 \circ 3) \circ 2 = \alpha(3) \ \Rightarrow\ \alpha(4) \circ 2 = \alpha(3) \ \Rightarrow\ 2 \circ 2 = 4 \ \Rightarrow\ 1 \neq 4 \\
{}^{l}\mathfrak{A} & \alpha(2 \circ 3) \circ \alpha(2) = 3 \ \Rightarrow\ \alpha(4) \circ \alpha(2) = 3 \ \Rightarrow\ 2 \circ 5 = 3 \ \Rightarrow\ 7 \neq 3 \\
{}^{r}\mathfrak{A} & (2 \circ 3) \circ \alpha(2) = \alpha(3) \ \Rightarrow\ 4 \circ \alpha(2) = \alpha(3) \ \Rightarrow\ 4 \circ 5 = 4 \ \Rightarrow\ 3 \neq 4 \\
{}^{sl}\mathfrak{A} & \alpha(2) \circ (3 \circ 2) = \alpha(3) \ \Rightarrow\ \alpha(2) \circ 5 = \alpha(3) \ \Rightarrow\ 5 \circ 5 = 4 \ \Rightarrow\ 1 \neq 4 \\
{}^{sr}\mathfrak{A} & \alpha(2) \circ \alpha(3 \circ 2) = 3 \ \Rightarrow\ \alpha(2) \circ \alpha(5) = 3 \ \Rightarrow\ 5 \circ 3 = 3 \ \Rightarrow\ 6 \neq 3
\end{array}
$$

Thus, in a WIP–quasigroup $(Q, \circ)$, only the identity of the variety $\mathfrak{A}$ holds for all $x, y \in Q$.

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2. Sokhatsky F. M., Lutsenko A. V., Fryz I. V. Construction of Quasigroups with Invertibility Properties. Journal of Mathematical Sciences, 2024, 279, 115 – 132.
3. Shcherbacov V. Elements of Quasigroup Theory and Applications. — Boca Raton: Chapman and Hall/CRC, 2017, 202 – 206.