Strongly consistent estimation of drift parameter in tempered fractional Ornstein–Uhlenbeck process

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Fractional Brownian motion is a well-known and widely used stochastic process. Its extensions – the tempered fractional Brownian motion (TFBM) and its second kind (TFBMII) – were introduced by Meerschaert and Sabzikar (see [1]), and by Sabzikar and Surgailis (see [2]), respectively.

TFBM $B^{I}_{H,\lambda} = \{B^{I}_{H,\lambda}(t), t \geq 0\}$ and TFBMII $B^{II}_{H,\lambda} = \{B^{II}_{H,\lambda}(t), t \geq 0\}$ are defined for positive values of H and λ, they are based on a two-sided Wiener process $\{W_s, s \in \mathbb{R}\}$ and have the following representations:

$$B^{I}_{H,\lambda}(t) := \int_{\mathbb{R}} \left[e^{-\lambda(t-s)_+}(t-s)_+^{H-\frac{1}{2}} - e^{-\lambda(-s)_+}(-s)_+^{H-\frac{1}{2}} \right] dW_s,$$

$$B^{II}_{H,\lambda}(t) := \int_{\mathbb{R}} \left[e^{-\lambda(t-s)_+}(t-s)_+^{H-\frac{1}{2}} - e^{-\lambda(-s)_+}(-s)_+^{H-\frac{1}{2}} + \lambda \int_0^t (u-s)_+^{H-\frac{1}{2}} e^{-\lambda(u-s)_+} du \right] dW_s.$$

They possess many properties that attract the attention of researchers due to their theoretical significance. TFBM and TFBMII can be used to model a wide range of real-world phenomena. Although these processes are relatively new, numerous results and equations involving them have already been studied. Nevertheless, many equations and models still require further investigation.

We study the behavior of the increments of TFBM and TFBMII. As a result, upper bounds for them were established. These conclusions are formulated and proved in the following theorems.

Theorem 1. *For any $\delta > 0$ and any $p > 2$ there exists a nonnegative random variable $\eta = \eta(\delta, p)$ such that for all $0 \leq t_2 < t_1 \leq t_2 + 1$*

$$\left| B^{I}_{H,\lambda}(t_1) - B^{I}_{H,\lambda}(t_2) \right| \leq \left(t_1^{\delta} \vee 1 \right) (t_1 - t_2)^{H \wedge 1} \left(|\log(t_1 - t_2)|^p \vee 1 \right) \eta \quad a.s.,$$

and there exist positive constants $C_1 = C_1(\delta, p)$ and $C_2 = C_2(\delta, p)$ such that for all $u > 0$

$$\mathsf{P}(\eta > u) \leq C_1 e^{-C_2 u^2}.$$

Theorem 2. *For any $\tau > 2$ and any $p > 2$ there exists a nonnegative random variable $\eta = \eta(\tau, p)$ such that for all $0 \leq t_2 < t_1 \leq t_2 + 1$*

$$\left| B^{II}_{H,\lambda}(t_1) - B^{II}_{H,\lambda}(t_2) \right| \leq \left(t_1^{\frac{1}{2}} (\log^+ t_1)^{\tau} \vee 1 \right) (t_1 - t_2)^{H \wedge \frac{1}{2}} \left(|\log(t_1 - t_2)|^p \vee 1 \right) \eta \quad a.s.,$$

and there exist positive constants $C_1 = C_1(\tau, p)$ and $C_2 = C_2(\tau, p)$ such that for all $u > 0$

$$\mathsf{P}(\eta > u) \leq C_1 e^{-C_2 u^2}.$$

As an application of these theorems, we study the problem of estimating the drift parameter in the following Langevin-type equation:

$$Y_t = y_0 + \theta \int_0^t Y_s ds + \sigma X_t, \quad t \geq 0,$$

where $y_0 \in \mathbb{R}$, θ and σ are positive parameters, and the driving process X is either TFBM $B^{I}_{H,\lambda}$ or TFBMII $B^{II}_{H,\lambda}$. The solution to such an equation is often referred to as the *tempered fractional Ornstein–Uhlenbeck process*.

We introduce and investigate estimators of the parameter $\theta > 0$, based on continuous or discrete observations of a trajectory of Y.

If Y is observed continuously over the entire interval $[0, T]$, then the estimator for θ is given by:

$$\widehat{\theta}_T = \frac{Y_T^2 - y_0^2}{2\int_0^T Y_t^2 dt}.$$

To obtain a discrete-time counterpart of the estimator $\widehat{\theta}_T$, we take a fixed $n > 1$, and set $T = n^{m-1}$, where $m > 1$. A trajectory of the process $\{Y_t, t \in [0, T]\}$ is observed at times k/n, $k = 0, \dots n^m$. Thus, $\{Y_{k/n}, k = 0, \dots, n^m\}$ is a discrete sample. In this case, the estimator for θ is the following:

$$\widetilde{\theta}_n(m) = \frac{n\left(Y^2_{n^{m-1}} - y_0^2\right)}{2\sum_{k=0}^{n^m-1} Y^2_{k/n}}.$$

We examine the behavior and asymptotic properties of $\widehat{\theta}_T$ and $\widetilde{\theta}_n(m)$. Our main results establish that these estimators are strongly consistent.

Theorem 3. *$\widehat{\theta}_T \to \theta$ a.s. as $T \to \infty$.*

Theorem 4. *$\widetilde{\theta}_n(m) \to \theta$ a.s. as $n \to \infty$.*

These theoretical results are supported by numerical simulations.

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1. Meerschaert M. M., Sabzikar F. Tempered fractional Brownian motion. Statistics & Probability Letters, 2013, 83, 10, 2269 – 2275.
2. Sabzikar F., Surgailis D. Tempered fractional Brownian and stable motions of second kind. Statistics & Probability Letters, 2018, 132, 17 – 27.