Orders of decay of entropy numbers and Kolmogorov width of mixed smoothness function classes

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We will present recent results on the exact-order estimates for some approximation characteristics of the Nikol'skii-Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ of periodic multivariate functions with mixed smoothness in the space $B_{q,1}$, $1 \le p, q \le \infty$, $1 \le \theta \le \infty$, whose norm is stronger than the L_q-norm. A motivation for investigating the approximation problems in this particular space $B_{q,1}$ is given by the fact that in L_q the respective exact-order estimates for some important cases still remain unknown (see Open problems in [1]). Considering these errors in the metric of a bit smaller space $B_{q,1}$ we achieved a significant progress over the known results in the Lebesque space.

Let $\mathscr{Y}$ be a normed space with the norm $\|\cdot\|_{\mathscr{Y}}$, $\mathscr{L}_M(\mathscr{Y})$ be the set of all subspaces of dimension at most M in the space $\mathscr{Y}$, and W be a centrally-symmetric set in $\mathscr{Y}$.

The quantity

$$d_M(W, \mathscr{Y}) := \inf_{L_M \in \mathscr{L}_M(\mathscr{Y})} \sup_{w \in W} \inf_{u \in L_M} \|w - u\|_{\mathscr{Y}}$$

is called the Kolmogorov M-width of the set W in the space $\mathscr{Y}$.

Let further $B_{\mathscr{Y}}(y, R)$ be a ball in $\mathscr{Y}$ of radius R centered at a point y, i.e.,

$$B_{\mathscr{Y}}(y, R) := \{x \in \mathscr{Y} : \ \|x - y\|_{\mathscr{Y}} \le R\}.$$

For a compact set A and $\varepsilon > 0$, we introduce the entropy numbers $\varepsilon_k(A, \mathscr{Y})$ as follows

$$\varepsilon_k(A, \mathscr{Y}) := \inf \Big\{ \varepsilon : \quad \exists y^1, \dots, y^{2^k} \in \mathscr{Y} : \quad A \subseteq \bigcup_{j=1}^{2^k} B_{\mathscr{Y}}(y^j, \varepsilon) \Big\}.$$

For positive quantities a and b, the notation $a \asymp b$ means that there exist positive constants C_1 and C_2, that do not depend on an essential parameter in the values of a, b, and such that $C_1 a \le b$ (in this case, we write $a \ll b$) and $C_2 a \ge b$ (denoted by $a \gg b$).

In what follows we formulate a statement concerning the lower estimate for the entropy numbers of the classes $B^{\boldsymbol{r}}_{\infty,\theta}$, $1 \le \theta \le \infty$, in the space $B_{1,1}$. It is an expansion of the corresponding result by V.N. Temlyakov of 1990 for the classes $H^{\boldsymbol{r}}_\infty \equiv B^{\boldsymbol{r}}_{\infty,\infty}$.

For the definition and properties of the Nikol'skii-Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ of functions with mixed smoothness as well as the norm of so-called "zero" Nikol'skii classes $B_{q,1}$ in which we measure the error, we refer to [1].

Theorem 1. [2] *Let $r_1 > 0$ and $1 \le \theta \le \infty$. Then it holds*

$$\varepsilon_M(B^{\boldsymbol{r}}_{\infty,\theta}, B_{1,1}) \gg M^{-r_1} (\log^{\nu-1} M)^{r_1 + 1 - \frac{1}{\theta}}.$$

For some parameter range we obtained also the matching upper bounds. Namely, the following exact order estimates hold.

Theorem 2. [2] *Let $d \geq 2$, $r_1 > 0$, $1 \leq q \leq p \leq \infty$, $1 \leq \theta \leq \infty$. Then it holds*

$$\varepsilon_M(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \asymp d_M(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1-\frac{1}{\theta}}. \tag{1}$$

We showed that the respective estimates for Kolmogorov widths are realized by the subspaces of trigonometric polynomials with harmonics from the step hyperbolic crosses of the respective dimension.

Besides we proved that in the multivariate case ($d \geq 2$) in contrast to the univariate ($d = 1$) in most of the considered situations the considered approximation characteristics of the classes $B^{\boldsymbol{r}}_{p,\theta}$ in the space $B_{q,1}$ differ in order from the corresponding characteristics in the space L_q.

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