Generalized symmetry algebra of Burgers equation

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Despite a number of relevant considerations in the literature, the algebra of generalized symmetries of the Burgers equation has not been exhaustively described. We fill this gap in [2], presenting a basis of this algebra in the most explicit form. We prefer to make a closed and simple proof from scratch, based on the relations between the (1+1)-dimensional (linear) heat equation $\mathcal{L}_1$, the potential Burgers equation $\mathcal{L}_2$ and the Burgers equation $\mathcal{L}_3$,

$$\mathcal{L}_1\colon\ u_t = u_{xx} \quad \overset{u=\mathrm{e}^w}{\longleftrightarrow} \quad \mathcal{L}_2\colon\ w_t = w_{xx} + w_x^2 \quad \overset{-2w_x=v}{\longleftrightarrow} \quad \mathcal{L}_3\colon\ v_t + vv_x = v_{xx},$$

which leads to the linearization of $\mathcal{L}_3$ to $\mathcal{L}_1$ by the Hopf–Cole transformation $v = -2u_x/u$. Another important ingredient is the exhaustive description of generalized symmetries of $\mathcal{L}_1$ in [1, Section 6]. The core of the proof is essentially simplified by using the original technique of choosing special coordinates in the associated jet space.

Below, instead of the total derivative operators with respect to t and x, we use their restrictions to the solution set of the corresponding equation $\mathcal{L}_i$,

$$\mathrm{D}_x := \partial_x + \sum_{k=0}^{\infty} z^i_{k+1}\partial_{z^i_k}, \quad \mathrm{D}_t := \partial_t + \sum_{k=0}^{\infty} \big(\mathrm{D}_x^k L^i[z^i]\big)\partial_{z^i_k},$$

where $L^1[u] := u_{xx}$, $L^2[w] := w_{xx} + w_x^2$, $L^3[v] := v_{xx} - vv_x$, $z^i_0 := z^i$, the jet variable z^i_k is identified with the derivative $\partial^k z^i/\partial x^k$, $k \in \mathbb{N}$, $z^1 := u$, $z^2 := w$ and $z^3 := v$.

Recall [1, Section 6] that the algebra of generalized symmetries of the (1+1)-dimensional linear heat equation $\mathcal{L}_1$ is $\Sigma_1 = \Lambda_1 \Subset \Sigma_1^{-\infty}$, where $\Lambda_1 := \big\langle \mathfrak{Q}^{kl},\, k, l \in \mathbb{N}_0\big\rangle$, $\Sigma_1^{-\infty} := \big\{\mathfrak{Z}(h)\big\}$ with $\mathfrak{Q}^{kl} := (\mathrm{G}^k\mathrm{P}^l u)\partial_u$, $\mathrm{P} := \mathrm{D}_x$, $\mathrm{G} := t\mathrm{D}_x + \frac12 x$, $\mathfrak{Z}(h) := h(t,x)\partial_u$, and the parameter function h runs through the solution set of $\mathcal{L}_1$. Elements of $\Sigma_1^{-\infty}$ are considered trivial generalized symmetries of $\mathcal{L}_1$ since in fact these are Lie symmetries of $\mathcal{L}_1$ that are associated with the linear superposition of solutions of $\mathcal{L}_1$. The complement subalgebra Λ_1 of $\Sigma_1^{-\infty}$ in Σ_1, which is constituted by the linear generalized symmetries of the equation $\mathcal{L}_1$, can be called the essential algebra of generalized symmetries of this equation. The algebra Λ_1 is generated by the two recursion operators P and G from the simplest linear generalized symmetry $u\partial_u$, and both the recursion operators and the seed symmetry are related to Lie symmetries.

Pulling back the elements of the algebra Σ_1 by the transformation $u = \mathrm{e}^w$, we obtain the algebra $\Sigma_2 = \Lambda_2 \Subset \Sigma_2^{-\infty}$ of generalized symmetries of the potential Burgers equation $\mathcal{L}_2$, which is thus isomorphic to the algebra Σ_1. As the counterparts of Λ_1 and $\Sigma_1^{-\infty}$, the subalgebra Λ_2 and the ideal $\Sigma_2^{-\infty}$ of Σ_2 are called the essential and the trivial algebras of generalized symmetries of $\mathcal{L}_2$, respectively.

Theorem 1. *The algebra of generalized symmetries of the Burgers equation $\mathcal{L}_3$ is*

$$\Sigma_3 := \big\langle \hat{\mathfrak{Q}}^{kl},\, (k,l) \in \mathbb{N}_0^2 \setminus \{(0,0)\}\big\rangle \quad \text{with} \quad \hat{\mathfrak{Q}}^{kl} := \big(\mathrm{D}_x\hat{\mathrm{G}}^k\hat{\mathrm{P}}^l 1\big)\partial_v,$$

where $\hat{\mathrm{P}} := \mathrm{D}_x - \frac12 v$ *and* $\hat{\mathrm{G}} := t\mathrm{D}_x + \frac12(x - vt)$.

Corollary 1. *The space of generalized symmetries of the Burgers equation* $\mathcal{L}_3$ *that are of order not greater than* n *is* $\Sigma_3^n := \big\langle \hat{\mathfrak{Q}}^{kl},\ (k,l)\in\mathbb{N}_0^2\backslash\{(0,0)\},\ k{+}l\leqslant n\big\rangle$, *and* $\dim\Sigma_3^n=\frac12 n(n{+}3)$.

Recall that the maximal Lie invariance algebra $\mathfrak{g}^{\rm B}$ of the equation $\mathcal{L}_3$ is five-dimensional, $\mathfrak{g}^{\rm B}=\langle\hat{\mathcal{P}}^t,\hat{\mathcal{D}},\hat{\mathcal{K}},\hat{\mathcal{P}}^x,\hat{\mathcal{G}}^x\rangle$, where

$$\hat{\mathcal{P}}^t=\partial_t,\quad \hat{\mathcal{D}}=2t\partial_t+x\partial_x-v\partial_v,\quad \hat{\mathcal{K}}=t^2\partial_t+tx\partial_x+(x-tv)\partial_v,\quad \hat{\mathcal{G}}^x=t\partial_x+\partial_v,\quad \hat{\mathcal{P}}^x=\partial_x.$$

In fact, the space $\Sigma_3^2=\big\langle\hat{\mathfrak{Q}}^{01},\,\hat{\mathfrak{Q}}^{10},\,\hat{\mathfrak{Q}}^{02},\,\hat{\mathfrak{Q}}^{11},\,\hat{\mathfrak{Q}}^{20}\big\rangle$ of generalized symmetries of the Burgers equation $\mathcal{L}_3$ that are of order not greater than 2 is closed with respect to Lie bracket of generalized vector fields, i.e., it is a five-dimensional Lie algebra. It is constituted by the canonical evolution forms of Lie symmetries of $\mathcal{L}_3$ and thus isomorphic to the algebra $\mathfrak{g}^{\rm B}$. More specifically, the basis elements $\hat{\mathcal{P}}^t$, $\hat{\mathcal{D}}$, $\hat{\mathcal{K}}$, $\hat{\mathcal{G}}^x$ and $\hat{\mathcal{P}}^x$ of $\mathfrak{g}^{\rm B}$ are associated, up to their signs, with the elements $2\hat{\mathfrak{Q}}^{02}$, $4\hat{\mathfrak{Q}}^{11}$, $2\hat{\mathfrak{Q}}^{20}$, $2\hat{\mathfrak{Q}}^{10}$ and $2\hat{\mathfrak{Q}}^{01}$ of Σ_3^2, respectively.

Corollary 2. *The characteristic of any generalized symmetry vector field of the Burgers equation* $\mathcal{L}_3$ *belongs to the image of* D_x.

Corollary 3. *The characteristic of any generalized symmetry vector field of the Burgers equation* $\mathcal{L}_3$ *of order* r *is a polynomial with respect to the jet variables up to order* r *and affine in* v_r, *where the coefficient of* v_r *is a polynomial with respect to* t *of degree not greater than* r, *and the degree of the characteristic with respect to* x *is less than or equal to* r.

Corollary 4. *The homomorphism* $\varphi\colon\Lambda_2\to\Sigma_3$ *of the algebra* Λ_2 *of essential generalized symmetries of the potential Burgers equation* $\mathcal{L}_2$ *to the entire algebra* Σ_3 *of generalized symmetries of the Burgers equation* $\mathcal{L}_3$, *which is induced by the differential substitution* $-2w_x=v$, *is an epimorphism, and* $\ker\varphi=\langle\tilde{\mathfrak{Q}}^{00}\rangle$.

Corollary 5. *The algebra* Σ_3 *of generalized symmetries of the Burgers equation* $\mathcal{L}_3$ *is isomorphic to the quotient algebra* $A_1(\mathbb{R})^{(-)}/\langle1\rangle$, *where* $A_1(\mathbb{R})^{(-)}$ *is the Lie algebra associated with the first Weyl algebra* $A_1(\mathbb{R})$, *and* $\langle1\rangle$ *is its center. Hence the algebra* Σ_3 *is simple, two-generated and, moreover, even one-and-half generated.*

Hence the commutation relations of the algebra Σ_3 are

$$[\hat{\mathfrak{Q}}^{kl},\hat{\mathfrak{Q}}^{k'l'}]=\sum_{i=1}^{\infty}i!\left(\binom{k'}{i}\binom{l}{i}-\binom{k}{i}\binom{l'}{i}\right)\hat{\mathfrak{Q}}^{k+k'-i,\,l+l'-i},$$

where $(k,l),(k',l')\in\mathbb{N}_0^2\setminus\{(0,0)\}$, and $\hat{\mathfrak{Q}}^{00}:=0$.

We also show that the two well-known recursion operators of the Burgers equation and its two seed generalized symmetries, which are evolution forms of its Lie symmetries, suffice to generate this algebra within the framework of the formal approach, whereas the zero generalized symmetry is sufficient as the only seed symmetry if the recursion operators are interpreted as Bäcklund transformations for the corresponding tangent bundle.

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2. Popovych D.R., Bihlo A. and Popovych R.O. Generalized symmetries of Burgers equation. 2024, arXiv:2406.02809.