Explosion and implosion of semi-Markov birth-death processes

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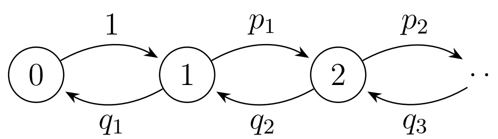
Consider a birth-death Markov chain $(X_k)_{k\geq 0}$ on the state space $\{0, 1, 2, \dots\}$ with transition probabilities $p_{i,i+1} = p_i$, $p_{i,i-1} = q_i$ such that $p_{0,1} = 1$ and p_i, $q_i > 0$, $p_i + q_i = 1$ for $i \geq 1$. Its diagram is given below.



Let $\{\tau_i\}_{i=0}^{\infty}$ be a sequence of positive random variables. For each τ_i consider a sequence $\{\tau_i^k\}_{k=0}^{\infty}$ of its independent copies. Suppose all these sequences and the Markov chain (X_k) are independent of each other. Put $T_0 = 0$ and define random moments of jumps recurrently:

$$T_{k+1} = T_k + \tau_{X_k}^k, \quad k \geq 0.$$

Definition 1. Let

$$X(t) = X_k, \quad t \in [T_k, T_{k+1}), \quad k \geq 0.$$

Then $X(t)$, $t \geq 0$ is a *semi-Markov* process with the embedded Markov chain (X_k) and waiting times $\{\tau_i\}$.

Let σ_n be the first moment when X hits $n \in \mathbb{N}$. Then $\sigma_\infty = \lim_{n\to\infty} \sigma_n$ denotes the time of hitting infinity.

Definition 2. The process X *explodes* (to infinity) if $\sigma_\infty < \infty$.

Let $\{Y_n\}_{n\geq 0}$ be a sequence of independent processes from Definition 1 such that $Y_n(0) = n$ for $n \geq 0$. Define stopping times

$$\theta_n = \inf\{t \geq 0 \mid Y_n(t) = n - 1\}, \quad n \geq 1.$$

Then $\Theta_\infty = \sum_{k=1}^{\infty} \theta_k$ represents the time of hitting 0 starting from infinity.

Definition 3. The process *implodes* (from infinity) if $\Theta_\infty < \infty$.

The following Theorems provide a generalization of known results about regularity for Markov birth-death processes (see [1, IV, §5]). They are also analogous to classical results on boundary classification for one-dimensional diffusions (see, e.g., [2, XV, §6]).

Introduce the following notation:

$$\delta_k = \frac{q_1 \cdots q_k}{p_1 \cdots p_k}, \quad k \geq 1,$$

$$\nu_i = (1 - \mathsf{E}e^{-\tau_i}) \frac{p_1 \cdots p_{i-1}}{q_1 \cdots q_i}, \quad i \geq 2,$$

and $\nu_0 = 1 - \mathsf{E}e^{-\tau_0}$, $\nu_1 = (1 - \mathsf{E}e^{-\tau_0}) \frac{1}{p_1}$.

Theorem 1 (Explosion condition)**.** *Let X be a semi-Markov process from Definition 1. For every initial distribution of X the following alternative holds:*

$$\mathsf{P}\{X \textit{ explodes}\} = 1 \iff \sum_{k=1}^{\infty}\left(\sum_{i=0}^{k}\nu_i\right)\delta_k < \infty,$$
$$\mathsf{P}\{X \textit{ explodes}\} = 0 \iff \sum_{k=1}^{\infty}\left(\sum_{i=0}^{k}\nu_i\right)\delta_k = \infty.$$

Theorem 2 (Implosion condition)**.** *Let X be a semi-Markov process from Definition 1. For every initial distribution of X the following alternative holds:*

$$\mathsf{P}\{X \textit{ implodes}\} = 1 \iff \sum_{k=1}^{\infty}\left(\sum_{i=k+1}^{\infty}\nu_i\right)\delta_k < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty}\delta_k = \infty,$$
$$\mathsf{P}\{X \textit{ implodes}\} = 0 \iff \sum_{k=1}^{\infty}\left(\sum_{i=k+1}^{\infty}\nu_i\right)\delta_k = \infty \quad \textit{or} \quad \sum_{k=1}^{\infty}\delta_k < \infty.$$

Now we apply Theorem 1 in the case when $\tau_i \overset{d}{=} \frac{\tau}{a_i}$, where a_i are some positive numbers and τ is a positive random variable.

Example 1. Suppose $\mathsf{E}\tau < \infty$. Then the necessary and sufficient condition for explosion is
$$\sum_{i=0}^{\infty}\frac{1}{a_i}\sum_{k=i}^{\infty}\frac{q_{i+1}\cdots q_k}{p_i\cdots p_k} < \infty.$$

Example 2. Suppose the distribution function F of τ is such that $1-F$ varies regularly at ∞ with exponent $-\alpha$, where $0<\alpha<1$. Then, assuming that $a_i \to \infty$ as $i \to \infty$, the necessary and sufficient condition for explosion is

$$\sum_{i=0}^{\infty}(1-F(a_i))\sum_{k=i}^{\infty}\frac{q_{i+1}\cdots q_k}{p_i\cdots p_k} < \infty.$$

1. Dynkin E. B., Yushkevich A. A. Markov Processes: Theorems and Problems. — New York: Springer New York, 1969, 237 p.
2. Karlin S., Taylor H. M. A Second Course in Stochastic Processes. — New York: Academic press, 1981, 542 p.