Open problems of weakly m-convex set theory

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Weakly m-convex sets, $m = 1, 2, \ldots, n-1$, in the n-dimensional real Euclidean space, $n \geqslant 2$, can be seen as a generalization of convex sets. Moreover, weakly m-convex sets can be disconnected, and under some additional conditions, the disconnectedness of a set in the space necessarily follows from its weak $(n-1)$-convexity. The theory of weakly m-convex sets is newish. The notion was coined by Yurii Zelinskii [1]. A special attention is given to the subclass of weakly m-convex sets to which correspond some sets of the complement to the entire space known as m-nonconvexity-point sets.

An open subset $E \subset \mathbb{R}^n$, $n \geqslant 2$, is called ***weakly*** m***-convex***, $m = 1, 2, \ldots, n-1$, if, through every boundary point of the set E, there passes an m-dimensional plane not intersecting E [1]. A closed subset $E \subset \mathbb{R}^n$, $n \geqslant 2$, is called ***weakly*** m***-convex***, $m = 1, 2, \ldots, n-1$, if it is approximated from the outside by a family of open weakly m-convex sets [1]. A point of the complement of a subset $E \subset \mathbb{R}^n$, $n \geqslant 2$, to the whole space $\mathbb{R}^n$ is an m***-nonconvexity point*** of the set E if every m-dimensional plane passing through the point intersects E. The collection of all m-nonsemiconvexity points of a subset $E \subset \mathbb{R}^n$ with respect to a fixed $m \in \{1, 2, \ldots, n-1\}$ is said to be the m***-nonconvexity-point set*** corresponding to E and is denoted by $E^{\triangle}_m$ [2].

Theorem 1 (Dakhil [3], Osipchuk [4])**.** *An open or a closed weakly $(n-1)$-convex subset $E \subset \mathbb{R}^n$ with non-empty $(n-1)$-nonconvexity-point set consists of not less than three connected components.*

Theorem 2 (Osipchuk [4])**.** *There exist weakly m-convex domains and closed connected subsets of $\mathbb{R}^n$ with non-empty m-nonconvexity-point sets, $m = 1, 2, \ldots, n-2$.*

Theorem 3 (Osipchuk [5, 6])**.** *Suppose that an open subset $E \subset \mathbb{R}^2$ is weakly 1-convex with non-empty set $E^{\triangle}_1$. Let $\left(E^{\triangle}_1\right)_j$, $j \in N \subseteq \mathbb{N}$, be the components of $E^{\triangle}_1$. Then*

(a) *$E^{\triangle}_1$ is open and weakly 1-convex;*

(b) *each component $\left(E^{\triangle}_1\right)_j$, $j \in N$, is the interior of a convex polygon, if E is bounded and consists of finite number of components.*

Theorem 4 (Osipchuk [7])**.** *Suppose that an open subset $E \subset \mathbb{R}^n$ is weakly 1-convex with non-empty set $E^{\triangle}_1$. Then $E^{\triangle}_1$ is open and weakly 1-convex, and there exists a collection of straight lines $\{\gamma(x)\}_{x \in \partial E^{\triangle}_1}$ passing through the points $x \in \partial E^{\triangle}_1$ such that*

- *the set $\bigcup\limits_{x \in \partial E^{\triangle}_1} \gamma(x) \bigcup E^{\triangle}_1$ does not contain straight lines passing through $E^{\triangle}_1$,*
- $\bigcup\limits_{x \in \partial E^{\triangle}_1} \gamma(x) \bigcap E = \varnothing$.

Theorem 5 (Osipchuk [7])**.** *There exists an open weakly 1-convex domain $E \subset \mathbb{R}^n$, $n \geqslant 3$, such that $E^{\triangle}_1$ is non-empty, bounded (or unbounded), connected, and non-convex.*

Theorem 6 (Osipchuk [7]). *Let $E \subset \mathbb{R}^n$ be a closed subset such that $\operatorname{Int} E \neq \varnothing$. If E is weakly 1-convex, then $\operatorname{Int} E$ is weakly 1-convex.*

The most natural next questions to study might be the following.

Problem 1. Suppose that an open subset $E \subset \mathbb{R}^n$, $n \geqslant 3$, is weakly m-convex with non-empty set $E^{\triangle}_m$, $m = 2, 3, \dots, n-1$. Is $E^{\triangle}_m$ also open and weakly m-convex?

Problem 2. Suppose that a closed subset $E \subset \mathbb{R}^n$, $n \geqslant 2$, is weakly m-convex with non-empty set $E^{\triangle}_m$, $m = 1, 2, \dots, n-1$. Is $E^{\triangle}_m$ also closed and weakly m-convex?

Problem 3. Suppose that a closed subset $E \subset \mathbb{R}^n$, $n \geqslant 3$, is weakly $(n-1)$-convex with non-empty set $E^{\triangle}_{n-1}$. Is each component of $E^{\triangle}_{n-1}$ convex?

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