Centralizers of Jacobian derivations with monomial coefficients

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Let K be an algebraically closed field of characteristic zero and $A = K[x, y]$ the polynomial ring in two variables. Recall that a K-derivation on A is a K-linear map $\mathcal{D} : A \longrightarrow A$ such that $\mathcal{D}(fg) = \mathcal{D}(f)\, g + f\, \mathcal{D}(g)$ for all $f, g \in A$. All the derivations on $A = K[x, y]$ form the Lie algebra $W_2(K)$ relative to the commutation, i.e., $[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1$. Every polynomial $f \in A$ defines the Jacobian derivation $\mathcal{D}_f$ on A by the rule:

$$D_f(g) = \det J(f, g),$$

where $J(f, g)$ is the Jacobi matrix of the polynomials f, g.

We study centralizers of Jacobian derivations of a special form in $W_2(K)$. The structure of such centralizers is of great interest, as, from a geometric point of view, every derivation $\mathcal{D} \in W_2(K)$ is a polynomial vector field on K^2.

Using some results from [2] and [1] we have proved the following statement which can can be useful while studying autonomous systems of differential equations (see, for example [3]):

Theorem 1. *Let $\mathcal{D}$ be a nonzero monomial Jacobian derivation of the form*

$$\mathcal{D} = -\alpha\,(n+1)\,x^{k+1}y^n\frac{\partial}{\partial x} + \alpha\,(k+1)\,x^k y^{n+1}\frac{\partial}{\partial y}, \quad \alpha \in K^*,\ k, n \geq 2.$$

Then

(1) If $k \neq n$, then $C_{W_2(K)}(\mathcal{D})$ is a free module of rank 2 over $\ker \mathcal{D}$ *with free generators $\mathcal{D}_1$ and T, where*

$$\mathcal{D}_1 = -\,n_1\,x\frac{\partial}{\partial x} + k_1\,y\frac{\partial}{\partial y} \text{ with}$$

$$k_1 = \frac{k+1}{d},\ n_1 = \frac{n+1}{d},\ d = \gcd(n+1, k+1),$$

$$T = -\frac{n}{k-n}\,x\frac{\partial}{\partial x} + \frac{k}{k-n}\,y\frac{\partial}{\partial y}.$$

(2) If $k = n \neq 0$, then $C_{W_2(K)}(\mathcal{D})$ has rank 1 over $\ker \mathcal{D}$ *with the free generator $\mathcal{D}_{xy}$.*

(3) If $k = n = 0$, then $C_{W_2(K)}(\mathcal{D})$ has rank 2 over $\ker \mathcal{D}$ *with the free generators*

$$\mathcal{D} = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad T = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

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