Geometric invariants of a random link

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In the work [1] of A.A.Dorogovtstev the construction of a random knot was proposed. The main advantage of the proposed construction is that the obtained random knot is smooth. This gives an opportunity to study geometrical properties of obtained knot using tools of analysis. Let us briefly recall the construction.

Let $\vec{\xi} = (\xi_1, \xi_2, \xi_3) : \mathbb{R}^2 \to \mathbb{R}^3$ be a centered Gaussian random field with independent identically distributed coordinates with the covariance

$$\mathbb{E}\xi_1(\vec{v})\xi_1(\vec{u}) = e^{-\frac{1}{2}||\vec{u}-\vec{v}||^2} \tag{1}$$

Suppose that θ is a point on a unit circle in plane. In [1] it was proved, that $\gamma_0(\theta) = \vec{\xi}(\theta)$ is a smooth random knot. By small adjustments of the proof given in [1] we can consider the construction of random link with two components. Our construction is the following. Suppose that θ_1, θ_2 are non-intersecting unit circles in plane and $\gamma_1 = \vec{\xi}(\theta_1), \gamma_2 = \vec{\xi}(\theta_2)$.

Theorem 1. *$\gamma = \gamma_1 \cup \gamma_2$ is a smooth random link. In other words, with probability 1 γ_1 and γ_2 are smooth non-intersecting curves in space.*

Now, we will define two geometric invariants of a link, linking number and average crossing number. For a link γ consisting of two components γ_1, γ_2 linking number $link(\gamma_1, \gamma_2)$ is defined by a well known Gauss linking integral [2]

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{(\dot{\gamma_1}(t), \dot{\gamma_2}(s), \gamma_1(t) - \gamma_2(s))}{||\gamma_1(t) - \gamma_2(s)||^3}\, dt\, ds \tag{2}$$

Another representation of a linking number is the following. Consider the projection of a link on a plane with at most finitely many transversal double points such that the orientation of a projection is agreed with the orientation of a link and such that the two paths at each double point are assigned to be the over path and the under path respectively. Every such double point, that created by arcs of two different components, we will call crossing and the projection we will call diagram of a link. Suppose that double points with over path and under path have signs 1 and -1 respectively. Then half of the sum of signs of crossings in the obtained diagram is exactly the linking number. The proof of equivalence of these two definitions and other definitions of linking number can be found in [2].

In [3] the definition of average crossing number of a link was proposed. Suppose that link γ consists of two components γ_1, γ_2, $\mathbb{S}^2$ is a unit sphere in $\mathbb{R}^3$. For $\vec{p} \in \mathbb{S}^2$ let $n(\gamma, \vec{p})$ be the number of crossings between γ_1 and γ_2 in a diagram of a link, when link γ is orthogonally projected on some plane, which is orthogonal to $\vec{p}$. Then average crossing number $acn(\gamma_1, \gamma_2)$ is equal to

$$\frac{1}{4\pi}\iint\limits_{\vec{p}\in\mathbb{S}^2} n(\gamma, \vec{p})\, dS \tag{3}$$

In [3] it was proved that average crossing number of a link has a representation similar to Gauss linking integral:

$$acn(\gamma_1, \gamma_2) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{|(\dot{\gamma_1}(t), \dot{\gamma_2}(s), \gamma_1(t) - \gamma_2(s))|}{||\gamma_1(t) - \gamma_2(s)||^3}\, dt\, ds \tag{4}$$

In our work we study properties of $acn(\gamma_1, \gamma_2), link(\gamma_1, \gamma_2)$ for previously defined random link. Firstly, we get some properties of distributions.

Proposition 1. *$link(\gamma_1, \gamma_2)$ has symmetric distribution. In other words*

$$\forall\, k \in \mathbb{Z} : \mathrm{P}\{link(\gamma_1, \gamma_2) = k\} = \mathrm{P}\{link(\gamma_1, \gamma_2) = -k\}$$

Before the next statement we propose the following property of average crossing number.

Proposition 2. *Suppose that deterministic link γ consists of two components γ_1, γ_2 and $acn(\gamma_1, \gamma_2) = 0$. Then γ_1 and γ_2 are non-intersecting trivial knots on some plane in $\mathbb{R}^3$.*

Using it we can prove the following property of average crossing number.

Theorem 2. *For a random link $\gamma = \gamma_1 \cup \gamma_2$*

$$\mathrm{P}\{acn(\gamma_1, \gamma_2) = 0\} = 0$$

Now we consider the existence of moments of these invariants.

Theorem 3. *For any $p \in [0, \frac{3}{2})$*

$$\mathbb{E}(acn(\gamma_1, \gamma_2))^p < \infty.$$

Corollary 1. *For any $p \in [0, \frac{3}{2})$*

$$\mathbb{E}|link(\gamma_1, \gamma_2)|^p < \infty.$$

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