Smooth functions on Klein bottle and homotopy types of orbits

B. Mazhar[1]**, S. Maksymenko**[1]

[1]Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

mazhar@imath.kiev.ua, maks@imath.kiev.ua

Let surface be a connected C^∞-smooth manifold of dimension 2, and let M be a compact surface. If $X \subset M$ is a closed subset, then define $\mathcal{D}(M,X) = \{h \in \mathcal{D}(M) \mid h|_X = \mathrm{id}|_X\}$ to be the group of diffeomorphisms of M, which are identity on X. If X is empty, we omit it. K denotes Klein bottle.

We study the homotopy types of orbits and stabilizers of the right action on the space of smooth functions $C^\infty(M,\mathbb{R}) \times \mathcal{D}(M,X) \to C^\infty(M,\mathbb{R})$ defined by precomposition $(f,h) \mapsto f \circ h$. Consider the class of smooth functions $\mathcal{F}(M,\mathbb{R})$ determined by the two properties:

1. Germs of the function f in every critical point $z \in M$ of f are smoothly equivalent to the germs of nonzero homogeneous polynomials $g_z \colon \mathbb{R}^2 \to \mathbb{R}$ without multiple factors.
2. Function f takes constant values on every component of the boundary ∂M and has no critical points on it.

Class $\mathcal{F}(M,\mathbb{R})$ contains a set of all Morse functions on surface M, which is dense and open in $C^\infty(M,\mathbb{R})$ considering strong Whitney topology; this makes the choice of class reasonable.

For each $f \in \mathcal{F}(M,\mathbb{R})$, let $\mathcal{S}(f,X)$ and $\mathcal{O}(f,X)$ be the stabilizer and the orbit of f with respect to the action defined above. Let $\mathcal{D}_{\mathrm{id}}(M,X)$, $\mathcal{S}_{\mathrm{id}}(f,X)$, and $\mathcal{O}_f(f,X)$ be connected components of $\mathcal{D}(M,X)$, $\mathcal{S}(f,X)$, $\mathcal{O}(f,X)$, respectfully containing denoted subscripts.

Proposition 1. *Let K be Klein bottle, $f \in \mathcal{F}(K,\mathbb{R})$. Then*

(a) *its Kronrod–Reeb graph Γ_f is acyclic, and there exists component α of some critical level set of f and open disks $D_1,\dots,D_m$ such that $K \setminus \alpha = \bigsqcup_{i=1}^m D_i$,*

(b) *its Kronrod–Reeb graph Γ_f is acyclic, and there exists component β of some regular level set of f and open Möbius bands M_1, M_2 such that $K \setminus \beta = M_1 \bigsqcup M_2$,*

(c) *or its Kronrod–Reeb graph Γ_f has a cycle, and there exists component C of some regular level set of f, corresponding to a point on an edge of the cycle, and open cylinders $Q_1,\dots,Q_m$ such that $K \setminus \{h(C) \mid h \in \mathcal{S}(f)\} = \bigsqcup_{i=1}^m Q_i$.*

Theorem 1. *If case* (b) from Proposition 1 holds, then there is an isomorphism

$$\pi_1\mathcal{O}_f(f) \cong \pi_1\mathcal{O}(f|_{M_1},\partial M_1) \times \pi_1\mathcal{O}(f|_{M_2},\partial M_2).$$

For Mobius band M group $\pi_1\mathcal{O}(f|_M,\partial M)$ was computed in [1].

Let $C \subset K$ be a closed curve, that corresponds to a point on an edge of the cycle of Γ_f. Let Q be the cylinder bounded by C and the next curve among $\{C_1 \equiv C, C_2,\dots,C_m\} = \{h(C) \mid h \in \mathcal{S}(f)\}$. Denote $G = \pi_1\mathcal{O}(f|_Q,\partial Q)$.

Theorem 2. *If case* (c) from Proposition 1 holds, then there are two possibilities:

(i) either for every $h \in \mathcal{S}(f)$ equality $h(C) = C$ implies that h preserves orientation of C, then m can be only odd, and there is an isomorphism

$$\pi_1\mathcal{O}_f(f) \cong G \wr_m \mathbb{Z},$$

(ii) or there exists $h \in \mathcal{S}(f)$ such that $h(C) = C$ and h changes orientation of C, then exists an automorphism $\gamma\colon G \to G$ with $\gamma^2 = \mathrm{id}$ and certain type of wreath product $\wr_{m,\gamma}$ depending on it such that there is an isomorphism

$$\pi_1 \mathcal{O}_f(f) \cong G \wr_{m,\gamma} \mathbb{Z}.$$

Theorem 3. *Consider composition $T^2 \xrightarrow{\pi} K \xrightarrow{f} \mathbb{R}$, where $f \in \mathcal{F}(M, \mathbb{R})$, and π is the orientable double covering of Klein bottle with the torus. Let $\mathcal{S}'(f, X) = \mathcal{D}_{id}(M, X) \cap \mathcal{S}(f, X)$. Then there are subgroups*

$$\pi_0 \mathcal{S}'(f) \hookrightarrow \pi_0 \mathcal{S}'(f \circ \pi), \qquad \pi_1 \mathcal{O}_f(f) \hookrightarrow \pi_1 \mathcal{O}_{f\circ\pi}(f \circ \pi).$$

Groups of type $\pi_0 \mathcal{S}'(f \circ \pi)$ and $\pi_1 \mathcal{O}_{f\circ\pi}(f \circ \pi)$ were computed in [2].

1. Kuznietsova I., Maksymenko S. Deformational symmetries of smooth functions on non-orientable surfaces. Topol. Methods Nonlinear Anal, 2025, 1–43.
2. Maksymenko S., Feshchenko B. Smooth functions on 2-torus whose Kronrod-Reeb graph contains a cycle. Methods Funct. Anal. Topology, 2015, 21, no. 1, 22–40.