VARIANT HARDY INEQUALITIES AND ITS RECENT IMPROVEMENTS

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Suppose that $p > 1$, and $a = \{a_n\}$ is a sequence of complex numbers such that $a \in \ell_p$, the Banach space of p-summable sequence spaces. Then the classical discrete Hardy's inequality states that

$$\sum_{n=1}^{\infty} \Big| \frac{1}{n} \sum_{k=1}^{n} a_k \Big|^p < \Big(\frac{p}{p-1} \Big)^p \sum_{n=1}^{\infty} |a_n|^p, \tag{1}$$

holds unless a_n is null ([2], Theorem 326). Also the constant term $C_p = \left(\frac{p}{p-1}\right)^p$ is sharp.

For $p = 2$, Hardy [1] first established the sharp dual inequality of (1) as below

$$\sum_{n=1}^{\infty} \Big| \sum_{k=n}^{\infty} \frac{a_k}{k} \Big|^2 \leq 4 \sum_{n=1}^{\infty} |a_n|^2, \tag{2}$$

where equality holds when all a_n are null. Copson [3] by adapting Elliott's proof and dual Hardy's inequality (2) introduced and studied a general variant Hardy's inequality as below

$$\sum_{n=1}^{\infty} q_n \Big| \sum_{k=n}^{\infty} \frac{q_k a_k}{Q_k} \Big|^2 \leq 4 \sum_{n=1}^{\infty} q_n |a_n|^2, \tag{3}$$

where the associated constant term is best possible. An equivalent form of the inequality (3) is reads as below

$$\sum_{n=2}^{\infty} \frac{Q_{n-1}^2}{q_{n-1}} |A_n - A_{n-1}|^2 \geq \frac{1}{4} \sum_{n=2}^{\infty} q_n |A_n|^2, \tag{4}$$

with $A_0 = A_1 = 0$. If we put $q_n = 1$ for each $n \in \mathbb{N}$ then one obtains an equivalent version of (2) as given below

$$\sum_{n=2}^{\infty} (n-1)^2 |A_n - A_{n-1}|^2 \geq \sum_{n=2}^{\infty} \frac{1}{4} |A_n|^2. \tag{5}$$

Similarly for $q_n = n, n^2, n^3$ for each $n \in \mathbb{N}$, then we get the reduced forms of (4) as below

$$\sum_{n=2}^{\infty} (n-1) n^2 |A_n - A_{n-1}|^2 \geq \sum_{n=2}^{\infty} n |A_n|^2, \tag{6}$$

$$\sum_{n=2}^{\infty} n^2 (2n-1)^2 |A_n - A_{n-1}|^2 \geq 9 \sum_{n=2}^{\infty} n^2 |A_n|^2, \tag{7}$$

$$\sum_{n=2}^{\infty}(n-1)n^4|A_n - A_{n-1}|^2 \geq 4\sum_{n=2}^{\infty} n^3|A_n|^2, \tag{8}$$

respectively. Improvement of the inequality (1) is being studied by Keller et al. [4]. The study of improvement of the variant Hardy inequalities (5-8) were remain to study. In this presentation, we discuss the improvement of the inequalities (5-8). For example, we prove the following result as an improvement of the inequality (5). Similarly, we present the improvement of other variant Hardy inequalities (6-8) in this presentation.

Theorem 1. *Suppose that $A = \{A_n\}$ be any sequence of complex numbers such that $A \in C_c(\mathbb{N}_0)$ with $A_0 = A_1 = 0$. Then*

$$\sum_{n=2}^{\infty}(n-1)^2|A_n - A_{n-1}|^2 \geq \sum_{n=2}^{\infty}\beta_n|A_n|^2 > \sum_{n=2}^{\infty}\frac{1}{4}|A_n|^2 \quad \textit{holds}$$

where the improved weight β_n for $n \geq 2$ is defined as below

$$\beta_n = n^2\Big[1 + \Big(1 - \frac{1}{n}\Big)^2 - \Big(1 + \frac{1}{n}\Big)^{-\frac{1}{2}} - \Big(1 - \frac{1}{n}\Big)^{\frac{3}{2}}\Big].$$

The presentation of the results is based on author's recent work published in [5].

1. Hardy G. H., Notes on some points in the integral calculus, LI.On Hillbert's double-series theorem,and some connected theorems concerning the convergence of infinite series and integrals, Messenger of Math., 1919, 48, 107-112.
2. Hardy G.H., Littlewood, J. E., Pólya, G. Inequalities. 2nd Edition, Cambridge University Press, 1967.
3. Copson E. T., Note on series of positive terms, J. London Math. Soc., 1927, s1-2(1), 9-12 & 1928, s1-3(1), 49-51.
4. Keller M., Pinchover Y., Pogorzelski, F., An improved discrete Hardy inequality, Amer. Math. Monthly, 2018, 125(4), 347-350.
5. Das B., Chakraborty S.K., Sadhu R., Manna A., Improvement of some discrete Hardy inequalities with variants, Glasgow Math. J., 2025, 67(1), 114-130.