Equivalence groupoids of classes of Schrödinger equations with time-independent potentials

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The study of Lie symmetries for Schrödinger equations was started in the 1970s with the linear case, and such studies have been continued up to now for more complicated classes of linear Schrödinger equations, see [2, 3, 5] and references therein. The recent paper [3] was devoted to the study of transformational properties and to the group classification of the class $\mathcal{F}$ of $(1{+}n)$-dimensional ($n\geqslant 1$) linear Schrödinger equations with complex-valued potentials, which are of the form

$$i\psi_t+\psi_{aa}+V(t,x)\psi=0, \tag{1}$$

where t and $x=(x_1,\dots,x_n)$ are the real independent variables, ψ is the complex dependent variable and V is an arbitrary smooth complex-valued potential depending on t and x. Here and in what follows the indices a and b run from 1 to n, and we assume summation over repeated indices. The equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ and the equivalence group $G^{\sim}_{\mathcal{F}}$ of the class $\mathcal{F}$ were computed, and then it was shown that this class is uniformly semi-normalized with respect to the linear superposition of solutions. This is why the group classification of $\mathcal{F}$ reduces to the classification of specific low-dimensional subalgebras of the associated equivalence algebra, which is completely realized for the case $n=2$. The counterparts of the above results for the subclass $\mathcal{F}_{\mathbb{R}}$ of linear Schrödinger equations with real-valued potentials were derived. Earlier, the analogous results were obtained in [2] for the case $n=1$.

We consider the subclasses $\mathcal{F}'$ and $\mathcal{F}'_{\mathbb{R}}$ of the class $\mathcal{F}$ that consist of the equations of the form (1) with time-independent complex- and real-valued potentials, respectively. Based on the description of $\mathcal{G}^{\sim}_{\mathcal{F}}$, we construct the equivalence groups of the above subclasses and describe their equivalence groupoids via classifying the admissible transformations within these subclasses.

Theorem 1. *(i) The equivalence group $G^{\sim}_{\mathcal{F}'}$ of the class $\mathcal{F}'$ consists of the point transformations in the space with the coordinates (t,x,ψ,ψ^*,V,V^*) whose (t,x,V)-components are of the form*

$$\tilde t=\lambda_1t+\lambda_0,\quad \tilde x_a=|\lambda_1|^{1/2}O^{ab}x_b+\nu_a,\quad \tilde\psi=\mathrm{e}^{i\lambda_3t+i\lambda_2+\lambda_5t+\lambda_4}\hat\psi,\quad \tilde V=\frac{\hat V}{|\lambda_1|}+\frac{\lambda_3-i\lambda_5}{\lambda_1},$$

where λ_0, …, λ_5 and ν_a are real constants with $\lambda_1\neq0$, and $O=(O^{ab})$ is an arbitrary constant $n\times n$ orthogonal matrix.

(ii) The equivalence group $G^{\sim}_{\mathcal{F}'_{\mathbb{R}}}$ of the class $\mathcal{F}'_{\mathbb{R}}$ is singled out from the group $G^{\sim}_{\mathcal{F}'}$ by the constraint $\lambda_5=0$.

Theorem 2. *(i) A generating (up to the $G^{\sim}_{\mathcal{F}'}$-equivalence and the linear superposition of solutions) set of admissible transformations for the class $\mathcal{F}'$ is the union of the following families of admissible transformations $(V,\Phi,\tilde V)$:*

$$\mathcal{T}_{1F}:=\big(x_n^{-2}F([x_1:\ldots:x_n])-x_ax_a,\ \Phi_1,\ \tilde x_n^{-2}F([\tilde x_1:\ldots:\tilde x_n])\big)\,,$$
$$\Phi_1\colon\quad \tilde t=\frac12\tan2t,\ \ \tilde x_a=\frac{x_a}{\cos2t},\ \ \tilde\psi=|\cos2t|^{n/2}\mathrm{e}^{i\tan(2t)|\boldsymbol{x}|^2/2}\psi,$$

$$\mathcal{T}_{2F} := \big(x_n^{-2}F([x_1:\ldots:x_n]) + x_ax_a,\, \Phi_2,\, \tilde x_n^{-2}F([\tilde x_1:\ldots:\tilde x_n])\big)\,,$$
$$\Phi_2\colon\ \ \tilde t = \frac14\mathrm{e}^{4t},\ \ \tilde x_a = \mathrm{e}^{2t}x_a,\ \ \tilde\psi = \mathrm{e}^{i|\boldsymbol{x}|^2/2-nt}\psi,$$

$$\mathcal{T}_{3\alpha U} := \big(U(x_2,\ldots,x_n) + i\alpha x_1 + x_1,\, \Phi_{3\alpha},\, U(\tilde x_2,\ldots,\tilde x_n) + i\alpha\tilde x_1\big),$$
$$\Phi_{3\alpha}\colon\ \ \tilde t = t,\ \ \tilde x_1 = x_1 - t^2,\ \ \tilde x_a = x_a,\ a\neq 1,\ \ \tilde\psi = \mathrm{e}^{-itx_1+(i+\alpha)t^3/3}\psi,$$

$$\mathcal{T}_{4\alpha U\kappa} := \big(U(x_2,\ldots,x_n) + i\alpha x_1 - x_1^2,\, \Phi_{4\alpha\kappa},\, U(\tilde x_2,\ldots,\tilde x_n) + i\alpha\tilde x_1 - \tilde x_1^2\big),$$
$$\Phi_{4\alpha\kappa}\colon\ \ \tilde t = t,\ \ \tilde x_1 = x_1 + 2\kappa\cos 2t,\ \ \tilde x_a = x_a,\ a\neq 1,\ \ \tilde\psi = \mathrm{e}^{\kappa\sin t\,(-2ix_1-2i\kappa\cos 2t-\alpha)}\psi,$$

$$\mathcal{T}_{5\alpha U\kappa\nu} := \big(U(x_2,\ldots,x_n) + i\alpha x_1 + x_1^2,\, \Phi_{5\alpha\kappa\nu},\, U(\tilde x_2,\ldots,\tilde x_n) + i\alpha\tilde x_1 + \tilde x_1^2\big),$$
$$\Phi_{5\alpha\kappa\nu}\colon\ \ \tilde t = t,\ \ \tilde x_1 = x_1 + 2\kappa\mathrm{e}^{2t} + 2\nu\mathrm{e}^{-2t},\ \ \tilde x_a = x_a,\ {\scriptstyle a\neq 1},\ \ \tilde\psi = \mathrm{e}^{(\kappa\mathrm{e}^{2t}-\nu\mathrm{e}^{-2t})(2i(x_1+\kappa\mathrm{e}^{2t}+\nu\mathrm{e}^{-2t})-\alpha)}\psi,$$

where $[x_k:\ldots:x_n]$ *with* $k\in\{1,\ldots,n\}$ *denotes homogeneous coordinates in the projective space of dimension* $n-k$*,* F *is a general sufficiently smooth complex-valued function in this space with* $k=1$*,* U *is a general sufficiently smooth complex-valued function of* $(x_2,\ldots,x_n)$*,* $\alpha,\kappa,\nu\in\mathbb{R}$*,* $\kappa\neq 0$ *in the fourth family, and* $(\kappa,\nu)\neq(0,0)$ *in the fifth family. Moreover, in the last two families,* $\alpha\neq 0$ *or* $x_2U_2+\cdots+x_nU_n+2U\neq 4\epsilon(x_2^2+\cdots+x_n^2)$*, where* $\epsilon=-1$ *and* $\epsilon=1$ *for the fourth and fifth families, respectively.*

(ii) A generating (up to the $G^{\sim}_{\mathcal{F}'_{\mathbb{R}}}$*-equivalence and the linear superposition of solutions) set of admissible transformations for the class* $\mathcal{F}'_{\mathbb{R}}$ *is the union of the families* $\{\mathcal{T}_{1F}\}$*,* $\{\mathcal{T}_{2F}\}$*,* $\{\mathcal{T}_{30U}\}$*,* $\{\mathcal{T}_{40U\kappa}\}$ *and* $\{\mathcal{T}_{50U\kappa\nu}\}$*, where in addition to the above conditions, the functions* F *and* U *are real-valued,* $\kappa,\nu\in\mathbb{R}$*, and in the last two families, we necessarily have the above condition for* U*.*

Since the classes $\mathcal{F}'$ and $\mathcal{F}'_{\mathbb{R}}$ do not have good normalization properties, we use results of [2, 3] and the above theorems to exhaustively solve the group classification problems for these subclasses in space dimensions one and two up to the general point equivalence and up to the equivalences generated by the corresponding equivalence groups. We hope to solve the analogous problems in space dimension three using an original algebraic version of the method of furcate splitting, which was initially proposed and applied in [4] and formalized in [6]. This method has been extended and applied to the group classification of various classes of differential equations, see [1, 6] and references therein.

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