On actions of stabilizers of smooth functions on edges of their Reeb graphs

I. V. Kuznietsova[1], **Yu.Yu. Soroka**[2]

[1]Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

[2]Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

kuznietsova@imath.kiev.ua, sorokayulya@imath.kiev.ua

Let M be compact connected surface, and P be either a real line or a circle. Denote by $D(M, X)$ the group of diffeomorphisms of M fixed on a closed subset $X \subset M$. There is a natural right action of the group $D(M, X)$ on the space of smooth functions $C^\infty(M, P)$ defined by the following rule: $(h, f) \longmapsto f \circ h$, where $h \in D(M, X)$, $f \in C^\infty(M, P)$.
Thus the *stabilizer* of f with respect to the action

$$\mathcal{S}(f, X) = \{h \in \mathcal{D}(M, X) \mid f \circ \ h = f\}$$

consists of f-preserving diffeomorphisms of M. Let

$$\mathcal{O}(f, X) = \{f \circ h \mid h \in \mathcal{D}(M, X)\}$$

be the *orbit* of f under the action. If $X = \emptyset$, then we will omit X from notation.

Endow the spaces $D(M, X)$, $C^\infty(M, P)$ with Whitney C^∞-topologies, and their subspaces $\mathcal{S}(f, X)$, $\mathcal{O}(f, X)$ with induced ones. Denote by $\mathcal{D}_{id}(M, X)$ the identity path component of $\mathcal{D}(M, X)$ and let $\mathcal{S}'(f, X) = \mathcal{S}(f) \cap \mathcal{D}_{id}(M, X)$.

Definition 1. Denote by $\mathcal{F}(M, P)$ the space of smooth maps $f \in C^\infty(M, P)$ having the following properties:

1. the map f takes constant values at each connected component of ∂M and has no critical points on it;
2. for every critical point z of f there is a local presentation $f_z \colon \mathbb{R}^2 \to \mathbb{R}$ of f near z such that f_z is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

Homotopy types of stabilizers and orbits of Morse functions were calculated in a series of papers by Sergiy Maksymenko, Bohdan Feshchenko, Elena Kudryavtseva and others. Furthermore, precise algebraic structure of such groups for the case $M \neq S^2, T^2$ was described in [1]. In particular it was proved there that the following theorem holds.

Theorem 1. *Let M be a connected compact oriented surface except 2-sphere and 2-torus and let $f \in \mathcal{F}(M, P)$. Then $\pi_0\mathcal{S}'(f, \partial M) \in \mathcal{B}$, where $\mathcal{B}$ is a minimal class of groups satisfying the following conditions:*

1) $1 \in \mathcal{B}$;

2) *if $A, B \in \mathcal{B}$, then $A \times B \in \mathcal{B}$;*

3 *if $A \in \mathcal{B}$ and $n \geq 1$, then $A \wr_n \mathbb{Z} \in \mathcal{B}$.*

Note that a group G belongs to the class $\mathcal{B}$ iff G is obtained from trivial group by a finite number of operations $\times$, $\wr_n \mathbb{Z}$. It is easy to see that every group $G \in \mathcal{B}$ can be written as a word in the alphabet $\mathcal{A} = \{1, \mathbb{Z}, (,), \times, \wr_2, \wr_3, \wr_4, \dots\}$. We will call such word a *realization* of the group G in the alphabet $\mathcal{A}$.

Denote by $\beta_1(G)$ the number of symbols $\mathbb{Z}$ in the realization ω of group G. The number $\beta_1(G)$ is the rank of the center $Z(G)$ and the quotient-group $G/[G,G]$ as shown in [2, Theorem 1.1]). Note, this number depends only on the group G, not the presentation ω. Moreover, $\beta_1(G)$ is first Betti number of $\mathcal{O}(f)$.

Let $W \subset \partial X$ be a connected component of ∂X. Edge of Γ_f will be called *external* if it is incident to the vertex of Γ_f that is corresponding to a non-degenerate critical point of f or non-fixed boundary component of ∂M with respect to the action of $S^{'}(f, W)$. Otherwise, it will be called *internal*. Denote by $\sharp Orb_{int}(M, W)$ the number of orbits of the action of $S^{'}(f, W)$ on internal edges of Γ_f.

Theorem 2. *Let M be a disk D^2 or a cylinder $C = S^1 \times [0,1]$ and $f \in \mathcal{F}(M, P)$. Then*

$$\sharp Orb_{int}(M, W) = \beta_1(\pi_0 S^{'}(f, \partial M)),$$

where $W = \partial M$ if $M = D^2$ or $W = S^1 \times 0$ if M is a cylinder.

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2. Kuznietsova, I. V., Soroka, Y. Y. First Betti numbers of orbits of Morse functions on surfaces. Ukrainian Mathematical Journal, 2021, Volume 73, 203 – 229 p.