Log-Sobolev Inequalities for Stochastic Differential Equations with Interactions

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We are interested in families $(\mu_t)_{t\geq 0}$ of probability distributions solving the following stochastic differential equation on $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X(u,t) &= \int_{\mathbb{R}^d} \phi(X(u,t)-X(v,t))\mu_0(\mathrm{d}v)\mathrm{d}t + \sigma \mathrm{d}B_t \\ X(u,0) &= u \in \mathbb{R}^d \\ \mu_t &= \mu_0 \circ X^{-1}(\cdot,t), \quad t \geq 0 \end{cases} \tag{1}$$

where $\sigma > 0$, $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, μ_0 is an initial probability distribution, and B_t is a standard d-dimensional Brownian motion. This system describes the evolution of interacting particles [1].

Definition 1. A probability measure μ on $\mathbb{R}^d$ satisfies a log-Sobolev inequality (LSI) with constant $C > 0$ if for all $h \in \mathcal{C}_c^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} h^2 \,\mathrm{d}\mu = 1$,

$$\int_{\mathbb{R}^d} h^2 \ln(h^2)\,\mathrm{d}\mu \leq C \int_{\mathbb{R}^d} |\nabla h|^2 \,\mathrm{d}\mu. \tag{2}$$

Our goal is to determine suitable conditions under which the family $(\mu_t)_{t\geq 0}$ solving (1) satisfies a LSI, and to calculate corresponding constants C_t. The drift term in (1) can be written as $b(X(u,t),\mu_t) = \int_{\mathbb{R}^d} \phi(X(u,t)-z)\mu_t(\mathrm{d}z)$.

Lemma 1. *Assume that the function ϕ from* (1) *satisfies a one-sided Lipschitz condition: for some constant $L_\phi \in \mathbb{R}$,*

$$(\phi(y_1)-\phi(y_2))\cdot(y_1-y_2) \leq L_\phi |y_1-y_2|^2 \quad \text{for all } y_1, y_2 \in \mathbb{R}^d. \tag{3}$$

Then, for any $x_1, x_2 \in \mathbb{R}^d$ and any probability measure μ, the drift $b(x,\mu)$ satisfies:

$$(b(x_1,\mu)-b(x_2,\mu))\cdot(x_1-x_2) \leq L_\phi |x_1-x_2|^2. \tag{4}$$

Lemma 2. *If the condition* (3) *holds, then for $t \geq 0$*

$$|X(u,t)-X(v,t)|^2 \leq |u-v|^2 e^{2L_\phi t}. \tag{5}$$

Lemma 3. *Consider a flow on $\mathbb{R}$ defined by a stochastic differential equation with interactions $\mathrm{d}X(u,t) = b(X(u,t),\mu_t)\mathrm{d}t + \sigma \mathrm{d}B_t$, where $X(u,0) = u$, $b(x,\mu) = \int_{\mathbb{R}} \phi(x-z)\mu(\mathrm{d}z)$, $\mu_t = \mu_0 \circ X^{-1}(\cdot,t)$. If the initial law μ_0 satisfies a LSI with constant C_0, and ϕ' is bounded, $\sup_x |\phi'(x)| \leq L_\phi$, then the law μ_t at time t a.s. satisfies a LSI with constant $C_t = C_0 e^{2L_\phi t}$.*

1. Dorogovtsev A. A. Measure-valued Processes and Stochastic Flows. — Berlin, Boston: Walter de Gruyter GmbH, 2024, 228 p.