Lie symmetries of generic Kolmogorov backward equations with power diffusivity

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We consider the peculiar class $\mathcal{F}'$ of the Kolmogorov backward equations with power diffusivity

$$\mathcal{F}'_\beta\colon\quad u_t + xu_y = |x|^\beta u_{xx},$$

where β is a real parameter. In [2] we discovered many unexpected and surprising transformational properties of $\mathcal{F}'$. For example, the class $\mathcal{F}'$ admits a distinguished discrete equivalence transformation

$$\mathscr{J}\colon\quad \tilde t = y\operatorname{sgn}x,\quad \tilde x = \frac{1}{x},\quad \tilde y = t\operatorname{sgn}x,\quad \tilde u = \frac{u}{x},\quad \tilde\beta = 5-\beta.$$

It turns out to be the only point equivalence transformation that is essential for the group classification of the class $\mathcal{F}'$, which was exhaustively carried out in [2].

Each equation $\mathcal{F}'_\beta$ admits the Lie invariance algebra $\mathfrak{g}_\beta$ spanned by the vector fields

$$\mathcal{P}^t := \partial_t,\ \mathcal{P}^y := \partial_y,\ \mathcal{I} := u\partial_u,\ \mathcal{D}^\beta := (2-\beta)t\partial_t + x\partial_x + (3-\beta)y\partial_y,\ \mathcal{Z}(f^\beta) := f^\beta\partial_u,$$

where the parameter function $f^\beta = f^\beta(t,x,y)$ runs through the solution set of this equation. In fact, the algebra $\mathfrak{g}_\beta$ is the maximal Lie invariance algebra of $\mathcal{F}'_\beta$ provided $\beta\in\mathbb{R}\setminus\{0,2,3,5\}$. The equations $\mathcal{F}'_0$ and $\mathcal{F}'_2$ are the so-called *remarkable Fokker–Planck* and *fine Kolmogorov backward* equations. The extended symmetry analysis of these equations was performed in [1, 3]. The equivalence transformation $\mathscr{J}$ maps the equations $\mathcal{F}'_0$ and $\mathcal{F}'_2$ to the equations $\mathcal{F}'_5$ and $\mathcal{F}'_3$, respectively.

In this talk, we discuss the extended symmetry analysis of the generic Kolmogorov backward equations from $\mathcal{F}'$, i.e., the equations $\mathcal{F}'_\beta$ with $\beta\in\mathbb{R}\setminus\{0,2,3,5\}$.

The vector fields $\mathcal{Z}(f^\beta)$ constitute the infinite-dimensional abelian ideal $\mathfrak{g}^{\rm lin}_\beta$ of the algebra $\mathfrak{g}^\beta$. This ideal is associated with the linear superposition of solutions of $\mathcal{F}'_\beta$. The complementary subspace $\mathfrak{g}^{\rm ess}_\beta$ to $\mathfrak{g}^{\rm lin}$ in $\mathfrak{g}^\beta$ that is spanned by the vector fields $\mathcal{P}^t$, $\mathcal{P}^y$, $\mathcal{I}$ and $\mathcal{D}^\beta$ is a subalgebra of $\mathfrak{g}^\beta$. Thus, the algebra $\mathfrak{g}^\beta$ splits over the ideal $\mathfrak{g}^{\rm lin}$. Up to the skew-symmetry of the Lie bracket of vector fields, the only nonzero commutation relations among the basis elements of $\mathfrak{g}^{\rm ess}_\beta$ are the following:

$$[\mathcal{P}^t,\mathcal{D}^\beta] = (2-\beta)\mathcal{P}^t,\quad [\mathcal{P}^y,\mathcal{D}^\beta] = (3-\beta)\mathcal{P}^y.$$

The algebra $\mathfrak{g}^{\rm ess}_\beta$ is isomorphic to the algebra $A^a_{3.4}\oplus A_1$ with $a=(2-\beta)/(3-\beta)$, see [6] for notation, which is consistent with Mubarakzyanov's algebra numeration [4].

Using the original combination of the direct method and the automorphism-based version of the algebraic method, we compute the point-symmetry pseudogroup G_β of $\mathcal{F}'_\beta$.

Theorem 1. *(i) For $\beta \in \mathbb{R} \setminus \{0, 2, 5/2, 3, 5\}$, the point-symmetry pseudogroup G_β of the equation $\mathcal{F}'_\beta$ consists of the point transformations*

$$\tilde{t} = |\alpha|^{2-\beta} t + \lambda_0, \quad \tilde{x} = \alpha x, \quad \tilde{y} = \alpha|\alpha|^{2-\beta} y + \lambda_1, \quad \tilde{u} = \sigma u + f(t, x, y),$$

where α, λ_0,λ_1 and σ are arbitrary constants with $\alpha\sigma \neq 0$, and f is an arbitrary solution of $\mathcal{F}'_\beta$.

(ii) In comparison with the general case of β, the point-symmetry pseudogroup $G_{5/2}$ of the equation $\mathcal{F}'_{5/2}$ is extended by the point transformations

$$\tilde{t} = \operatorname{sgn}(x)\alpha|\alpha|^{2-\beta} y + \lambda_1, \quad \tilde{x} = \frac{\alpha}{x}, \quad \tilde{y} = \operatorname{sgn}(x)|\alpha|^{2-\beta} t + \lambda_0, \quad \tilde{u} = \sigma\frac{u}{x} + f(t, x, y).$$

For the purpose of Lie reductions of the equation $\mathcal{F}'_\beta$ for each value $\beta \in \mathbb{R}\setminus\{0, 2, 3, 5\}$, we classified the subalgebras of the algebra $\mathfrak{g}^{\rm ess}_\beta$ modulo the adjoint action of the pseudogroup G_β. This classification follows from the isomorphism between $\mathfrak{g}^{\rm ess}_\beta$ and $A^a_{3.4} \oplus A_1$ with $a = (2-\beta)/(3-\beta)$ and the related results of Patera and Winternitz from [5]. Based on it, we construct wide families of exact solutions of the equations $\mathcal{F}'_\beta$ with $\beta \in \mathbb{R} \setminus \{0, 2, 3, 5\}$, in particular those associated with the codimension-one Lie reductions to the linear $(1+1)$-dimensional heat equations with the zero or the inverse square potentials. The most prominent among them are the solutions associated with the value $\beta = 1$ and their $G^\sim_{\mathcal{F}'}$-counterparts with $\beta = 4$. Another prominent case is given by some codimension-two Lie reductions with $\beta = -1$ and their $G^\sim_{\mathcal{F}'}$-counterparts with $\beta = 6$, which results in Whittaker equations, see [2, Section 8].

Following the approach from [3], we discuss the solution generation for the generic Kolmogorov backward equations $\mathcal{F}'_\beta$ using the action by recursion operators of $\mathcal{F}'_\beta$ associated with Lie symmetries of $\mathcal{F}'_\beta$. It is particularly surprising that in the prominent cases $\beta = 1$ and $\beta = 4$, as well as $\beta = -1$ and $\beta = 6$, the constructed families of solutions can be significantly extended using the above action.

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