A law of the iterated logarithm for random Dirichlet series: boundary case

R. O. Kostohryz[1]

[1]Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

kostogriz2909@gmail.com

By random Dirichlet series, we mean a random series parameterized by $s > 0$:

$$X_\alpha(s) := \sum_{k\geq 2} \frac{(\log k)^\alpha}{k^{1/2+s}} \eta_k,$$

where $\alpha \in \mathbb{R}$ and $\eta_1, \eta_2, \dots$ are independent copies of a random variable η with zero mean and finite variance. In 2023, a law of the iterated logarithm (LIL) was proved for random Dirichlet series as $s \to 0+$, where $\alpha > -1/2$ (see [1, Theorem 3.1]). In [2], the authors obtained a LIL in a boundary case $\alpha = -1/2$.

For a family (x_t) of real numbers denote by $C((x_t))$ the set of its limit points.

Theorem 1. *Assume that $\mathbb{E}[\eta] = 0$ and $\sigma^2 = \mathbb{E}[\eta^2] \in (0, \infty)$. Then*

$$C\Big(\Big(\Big(\frac{1}{2\sigma^2 \log 1/s\ \log\log\log 1/s}\Big)^{1/2} \sum_{k\geq 2} \frac{(\log k)^{-1/2}}{k^{1/2+s}} \eta_k \ : \ s \ \in \ (0, \mathrm{e}^{-\mathrm{e}})\Big)\Big) \ = \ [-1, 1] \quad \text{a.s.}$$

The boundary case is more demanding technically. When moving from $\alpha > -1/2$ to $\alpha = -1/2$ there is a phase transition in the asymptotic of the variance of the random series, which consists in a change of the asymptotic from polynomial to logarithmic.

The proof of the LIL is divided into two parts. First, it is shown that the upper limit of the series, properly normalized, does not exceed an explicitly given constant. Then it is proven that the same limit is not smaller than the same constant. To obtain the former inequality the series was split into three fragments, and it was demonstrated that the contributions of two fragments are negligible. When analyzing the principal fragment it was proven that there is an almost sure convergence along a sequence, and then this convergence was extended to real numbers. The last part of this programme is most troublesome. Here, the argument uses a chaining argument, which is the modern approach to an investigation of maxima of continuous-time processes.

In the second part of the proof, a fragment of the series was singled out that gives a principal contribution. Finding such a fragment is highly nontrivial, for afterwards it should be treated with the help of the converse Borel-Cantelli lemma which requires independence.

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