Limit theorems for globally perturbed random walks

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Let (ξ_1, η_1), $(\xi_2, \eta_2), \ldots$ be independent copies of an $\mathbf{R}^2$-valued random vector (ξ, η) with arbitrarily dependent components. Denote by $(S_n)_{n\geq 0}$ the zero-delayed standard random walk with increments ξ_n for $n \in \mathbf{N} := \{1, 2, \ldots\}$, that is, $S_0 := 0$ and $S_n := \xi_1 + \ldots + \xi_n$ for $n \in \mathbf{N}$. Put

$$T_n := S_{n-1} + \eta_n, \quad n \in \mathbf{N}.$$

The sequence $T := (T_n)_{n\geq 1}$ is called *globally perturbed random walk*. Many results concerning T accumulated up to 2016 can be found in the book [1].

For $t \in \mathbf{R}$, define the *first passage time* into (t, ∞)

$$\tau(t) := \inf\{n \geq 1 : T_n > t\},$$

the *number of visits* to $(-\infty, t]$

$$N(t) := \sum_{n\geq 1} \mathbf{1}_{\{T_n \leq t\}}$$

and the associated *last exit time*

$$\rho(t) \ := \ \sup\{n \geq 1 : T_n \leq t\}.$$

We prove weak law of large numbers for $\tau(t)$ and strong laws of large numbers for $\tau(t)$, $N(t)$ and $\rho(t)$. There is a difference between the first order asymptotic behavior of $\tau(t)$ and that of $N(t)$ and $\rho(t)$. The former depends heavily upon the right distribution tail of η, whereas the latter does not depend on it at all.

Theorem 1. *Suppose $\mu = \mathrm{E}[\xi] \in (0, \infty)$. The following assertions are equivalent:*
*(*W1*)* $\lim_{t\to\infty} t^{-1}\tau(t) = \mu^{-1}$ *in probability;*
*(*W2*)* $\lim_{n\to\infty} n^{-1} \max_{1\leq k\leq n} T_k = \mu$ *in probability;*
*(*W3*)* $\lim_{t\to\infty} t\mathrm{P}\{\eta > t\} = 0$.

Theorem 2. *Suppose $\mu = \mathrm{E}[\xi] \in (0, \infty)$. The following assertions are equivalent:*
*(*S1*)* $\lim_{t\to\infty} t^{-1}\tau(t) = \mu^{-1}$ a.s.*;*
*(*S2*)* $\lim_{n\to\infty} n^{-1} \max_{1\leq k\leq n} T_k = \mu$ a.s.*;*
*(*S3*)* $\mathrm{E}[\eta^+] < \infty$.

Theorem 3. *Suppose $\mu = \mathrm{E}[\xi] \in (0, \infty)$. If, for some $t \in \mathrm{R}$, $N(t) < \infty$ a.s. or $\rho(t) < \infty$ a.s., then*

$$\lim_{t\to\infty} \frac{N(t)}{t} = \lim_{t\to\infty} \frac{\rho(t)}{t} = \frac{1}{\mu} \quad \text{a.s.}$$

For $a, b > 0$, let $(t_k^{(a,b)}, j_k^{(a,b)})$ be the atoms of a Poisson random measure $N^{(a,b)}$ on $[0, \infty) \times (0, \infty)$ with mean measure $\mathrm{LEB} \times \mu_{a,b}$, where LEB is Lebesgue measure on $[0, \infty)$ and $\mu_{a,b}$ is a measure on $(0, \infty]$ defined by

$$\mu_{a,b}((x, \infty]) = ax^{-b}, \quad x > 0.$$

Denote by $D := D[0,\infty)$ the Skorokhod space, that is, the set of càdlàg functions defined on $[0,\infty)$. We shall use the J_1- and M_1-topologies, which are standard topologies on D. We write $\Longrightarrow$ to denote weak convergence in a function space.

We prove two distributional limit theorems for $\tau(t)$ for some distributions of η.

Theorem 4. *Suppose $\mu = \mathrm{E}[\xi] \in (-\infty,\infty)$ and $\mathrm{P}\{\eta > t\} \sim c/t$ as $t \to \infty$ for some $c > 0$. Then*

$$\Big(\frac{\tau(ut)}{t}\Big)_{u\geq 0} \Longrightarrow \Big(\inf\{z \geq 0 : \max_{k:\, t_k^{(c,\,1)} \leq z} (\mu t_k^{(c,\,1)} + j_k^{(c,\,1)}) > u\}\Big)_{u\geq 0}, \quad t \to \infty$$

in the M_1-topology on D.

Theorem 5. *Suppose $\mu = \mathrm{E}[\xi] \in (-\infty,\infty)$ and $\mathrm{P}\{\eta > x\} \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ for some $\alpha \in (0,1)$ and some ℓ slowly varying at ∞. Then*

$$\big(\mathrm{P}\{\eta > t\}\tau(ut)\big)_{u\geq 0} \Longrightarrow \Big(\inf\{z \geq 0 : \max_{k:\, t_k^{(1,\,\alpha)} \leq z} j_k^{(1,\,\alpha)} > u\}\Big)_{u\geq 0}, \quad t \to \infty$$

in the M_1-topology on D.

We obtain several functional limit theorems which quantify the rate of convergence in the laws of large numbers for $\tau(t)$, $N(t)$ and $\rho(t)$.

Theorem 6. *Suppose $\mu = \mathrm{E}[\xi] \in (0,\infty)$ and $\sigma^2 := \mathrm{Var}\,[\xi] \in (0,\infty)$.*
If $\mathrm{E}[\eta^+] < \infty$, then

$$\Big(\frac{\tau(ut) - \mu^{-1}ut}{(\sigma^2\mu^{-3}t)^{1/2}}\Big)_{u\geq 0} \overset{\text{f.d.}}{\Longrightarrow} \big(B(u)\big)_{u\geq 0}, \quad t \to \infty,$$

where $(B(u))_{u\geq 0}$ is a standard Brownian motion.
If $\mathrm{E}[\eta^-] < \infty$, then

$$\Big(\frac{\rho(ut) - \mu^{-1}ut}{(\sigma^2\mu^{-3}t)^{1/2}}\Big)_{u\geq 0} \overset{\text{f.d.}}{\Longrightarrow} \big(B(u)\big)_{u\geq 0}, \quad t \to \infty.$$

If $\mathrm{E}[\eta] \in (-\infty,\infty)$, then

$$\bigg(\Big(\frac{\tau(ut) - \mu^{-1}ut}{(\sigma^2\mu^{-3}t)^{1/2}}\Big)_{u\geq 0}, \Big(\frac{N(ut) - \mu^{-1}ut}{(\sigma^2\mu^{-3}t)^{1/2}}\Big)_{u\geq 0}, \Big(\frac{\rho(ut) - \mu^{-1}ut}{(\sigma^2\mu^{-3}t)^{1/2}}\Big)_{u\geq 0}\bigg)$$
$$\overset{\text{f.d.}}{\Longrightarrow} \left((B(u))_{u\geq 0}, (B(u))_{u\geq 0}, (B(u))_{u\geq 0}\right), \quad t \to \infty.$$

Theorem 7. *Suppose $\mu = \mathrm{E}[\xi] \in (0,\infty)$, $\sigma^2 = \mathrm{Var}\,[\xi] \in (0,\infty)$ and $\mathrm{E}[\eta^-] < \infty$. Then*

$$\Big(\frac{N(ut) - \mu^{-1}ut + \mu^{-1}\int_0^{ut} \mathrm{P}\{\eta > y\}\mathrm{d}y}{(\sigma^2\mu^{-3}t)^{1/2}}\Big)_{u\geq 0} \Longrightarrow \big(B(u)\big)_{u\geq 0}, \quad t \to \infty$$

in the J_1-topology on D, where $(B(u))_{u\geq 0}$ is a standard Brownian motion.

The presentation is based on the article [2].

1. Iksanov A. Renewal theory for perturbed random walks and similar processes. — Cham: Birkhäuser, 2016, 250 p.
2. Iksanov A., Kondratenko O. Limit theorems for globally perturbed random walks. Stochastic Models, 2025 (to appear). Preprint available at https://arxiv.org/abs/2501.02123.