Upper-bound simulation method for a Brownian bridge with a given minimum

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Brownian bridge simulation methods are an important part of Monte Carlo techniques, particularly in the context of exotic option pricing problems where closed-form solutions are not available. In this case, two issues emerge: the time discretization of paths introduces a certain degree of bias into the estimates, and extremely small time increments may be required to obtain the desired accuracy with a high impact on computational efficiency (for an overview, see, for example, [1]). In order to eliminate the bias, Beskos et al. [2] presented the so-called ε-strong algorithm, which provides the two-sided bounds for the Brownian path.

The method involves the iterative completion of several stages: initialization (determining the initial boundaries for a certain time interval), bisection (sampling the midpoint), and refinement (controlling that the upper and lower boundaries for the minimum and maximum of the path are not greater than the square root of the interval length). The initialization stage can be simple, as the problem at hand may provide a set of natural boundaries. In general case, one can use the acceptance-rejection sampling with squeezing among a set of shifted intervals, which is given in [2, Section 5.3] (for the modified version, see [1, Algorithm 6.3.1]). We consider an alternative approach to sampling using the cdf inversion, which relies on the following direct consequence of the Doob's results [3, (4.3)].

Proposition 1. *Let $\left\{B_t^{x,l}, t\in[0,l]\right\}$ be the Brownian bridge process with value $x\in R^1$ at 0 and at $l>0$. Set $B_{x,l}^- = \inf_{t\in[0,l]} B_t^{x,l}$ and $B_{x,l}^+ = \sup_{t\in[0,l]} B_t^{x,l}$. The conditional cdf of the Brownian bridge maximum with given minimum $u\le x$ can be represented as the series*

$$F_{B_{x,l}^+|B_{x,l}^-=u}(v) = \begin{cases} 2\sum_{n=1}^{\infty} F_n^{x-u,l}(v-u), & v\ge x,\\ 0 & v<x,\end{cases} \tag{1}$$

where

$$F_n^{\Delta_0,l}(\Delta) = e^{-\frac{2n\Delta}{l}(n\Delta-2\Delta_0)}(n-1)(n\Delta-\Delta_0) + e^{-\frac{2(n-1)\Delta}{l}((n-1)\Delta+2\Delta_0)} n((n-1)\Delta+\Delta_0)$$
$$-e^{-\frac{2}{l}(n\Delta-\Delta_0)(n\Delta+\Delta_0)} 2n^2\Delta, \quad \Delta\ge\Delta_0\ge 0.$$

Remark 1. For relatively big l and small n the terms $F_n^{\Delta_0,l}(\Delta)$ are not necessarily non-decreasing non-negative functions of Δ. This complicates the application of formula (1) for determining the approximate inverse of the conditional cdf. By restricting the consideration to the domain for which the sum of the first N terms is non-negative, we can obtain an approximation from below for $F_{B_{x,l}^+|B_{x,l}^-=u}$, and consequently, an approximation from above for the maximum.

1. Pollock M. Some Monte Carlo Methods for Jump Diffusions. PhD Thesis, The University of Warwick, 2013, 263 p.
2. Beskos A., Peluchetti S., Roberts G. ε-strong simulation of the Brownian path. Bernoulli, 2012, 18, 1223 – 1248.
3. Doob J. L. Heuristic approach to the Kolmogorov-Smirnov theorems. The Annals of Mathematical Statistics, 1949, 20, 3, 393 – 403.