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Gradient-type Kuramoto system for frustrated spin system

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To investigate the collective behavior of coupled frustrated spins [1, 2, 3, 4], we propose a model based on a network of coupled phase oscillators. Specifically, we consider a ring network with competing interactions between nearest and next-nearest neighbors: the coupling is attractive or repulsive depending on the distance between elements. The initial stage of the study involves transitioning from a ring spin network, defined in a complex space, to a gradient-type Kuramoto system with a circulant coupling matrix.

Set $\Omega := [0,1)^2$ and consider the square lattice $\Omega \cap \varepsilon\mathbb{Z}^2$, $\varepsilon > 0$. To each atom $(i\varepsilon, j\varepsilon) \in \Omega \cap \varepsilon\mathbb{Z}^2$, we associate a spin vector $u_{i,j} \in \mathbb{S}^1$. The configurational energy of a spin field $u : \Omega \cap \varepsilon\mathbb{Z}^2 \to \mathbb{S}^1$, for an interaction parameter $\eta > 0$, is given by

$$I_{\eta,\varepsilon} := -\eta \sum_{(i,j)} u_{i,j} \cdot u_{i+1,j} - \eta \sum_{(i,j)} u_{i,j} \cdot u_{i,j+1} + \sum_{(i,j)} u_{i,j} \cdot u_{i+2,j} + \sum_{(i,j)} u_{i,j} \cdot u_{i,j+2}.$$

Consider the energy functional under periodic boundary conditions and reformulate it in terms of angular variables, representing each spin as $u_{k,l} := \exp(i\theta_{k,l})$. Focusing on horizontal interactions, the energy simplifies to

$$F_{\eta,\varepsilon} = \sum_{i \in \varepsilon\mathbb{Z} \cap [0,1)} \left| u_{i+2} - \frac{\eta}{2} u_{i+1} + u_i \right|^2 = \sum_{i \in \varepsilon\mathbb{Z} \cap [0,1)} \Big(2 + \frac{\eta^2}{4} - 2\eta \cos(\theta_{i+1} - \theta_i) + 2\cos(\theta_{i+2} - \theta_i) \Big).$$

We obtain the gradient system in the following form:

$$\dot{\theta}_i = a \sin(\theta_i - \theta_{i+1}) + a \sin(\theta_i - \theta_{i-1}) + b \sin(\theta_i - \theta_{i+2}) + b \sin(\theta_i - \theta_{i-2}), \tag{1}$$

with periodic boundary conditions $\theta_{N+i} = \theta_i$.

The system (1) exhibits dihedral symmetry $\mathbf{D}_N$ (rotations and reflections), which leads to the existence of invariant subspaces and symmetric families of solutions.

Proposition 1. *System (1) has an invariant manifold*

$$\mathcal{M} = \big\{ (\theta_1, \ldots, \theta_N) \colon \theta_i - \theta_{N-i},\ i = 1, \ldots, [(N-1)/2] \big\}.$$

In phase differences $\varphi_i = \theta_1 - \theta_{i+1}, i = 1, \ldots, N-1$, the system will take the form

$$\begin{aligned} \dot{\varphi}_i &= a \sin(\varphi_1) + a \sin(\varphi_{N-1}) + b \sin(\varphi_2) + b \sin(\varphi_{N-2}) \\ &\quad + a \sin(\varphi_i - \varphi_{i+1}) + a \sin(\varphi_i - \varphi_{i-1}) + b \sin(\varphi_i - \varphi_{i+2}) + b \sin(\varphi_i - \varphi_{i-2}). \end{aligned}$$

We identify three main types of equilibria for the system in phase differences that exist for all N and for arbitrary real parameters a, b:

- full synchronization $\Phi_{sync} = (\varphi_1, \varphi_2, \ldots, \varphi_{N-1}) = (0, 0, \ldots, 0)$;
- splay states $\Phi^k_{splay} = (\varphi_1, \varphi_2, \ldots, \varphi_{N-1}) = \big(k\frac{2\pi}{N}, 2k\frac{2\pi}{N}, \ldots, (N-1)k\frac{2\pi}{N}\big)$;

- π-states (anti-phase) $\Phi_\pi = (\varphi_1, \varphi_2, \dots, \varphi_{N-1}) = (\phi, \phi, \dots, \phi)$, $\phi \in \{0, \pi\}$.

For the splay states, we derive an explicit formula for the eigenvalues of the linearized system, which allows us to characterize their stability analytically:

$$\lambda_m \left(\Phi^k_{splay}\right) = \sum_{j=1}^{N-1} K_j g' \left(\frac{2\pi k}{N} j\right) \left(1 - e^{\imath \frac{2mj\pi}{N}}\right), \quad m = 1, \dots, N-1,$$

where in our case $g(x) = \sin(x), K_1 = K_{N-1} = K_{-1} = a$, $K_2 = K_{N-1} = K_{-2} = b$ and $K_i = 0$ for all other indices. This observation also applies to the fully synchronized states, since $\Phi^0_{splay} = \Phi_{sync}$. This expression describes the linear stability of the above equilibria and also indicates the parametric values of the degenerate bifurcations.

It is important to emphasize that the fixed points described above do not represent a comprehensive classification of the system's equilibria. The system admits additional equilibrium configurations associated with more intricate collective behaviors. The existence and structure of these equilibria generally depend on the number of oscillators N, as well as on the specific coupling parameters.

As a starting point, we analyze the simplest nontrivial case $N = 4$. In this setting, we identify new equilibria that do not fall into the standard classes described above. These new equilibria arise under specific parameter relations and exhibit symmetry properties distinct from the known types. Furthermore, the structure of these equilibria can be extended to arbitrary system sizes such that $N = 4k$, $k \in \mathbb{N}$.

Proposition 2. *For any coefficients a, b and $N = 4k$, $k \in \mathbb{N}$, states where $\theta_i = \theta_{i+2} + \pi$ are equilibria of the system (1).*

Proposition 3. *For any $a, b \in \mathbb{R}$ such that $a = -2b$ and $N = 4k$, $k \in \mathbb{N}$, states where $\theta_i = \theta_{i+1} = \theta_{i+4m} = \theta_{i+4m+1}$ are equilibria of the system (1).*

Future work will focus on extending the analysis to systems of arbitrary size N, including a detailed study of equilibrium structures and their bifurcations for both even and odd N. Another direction involves generalizing the model to two-dimensional spin networks with periodic boundary conditions, reducing them to corresponding gradient oscillator systems, and exploring their equilibrium and dynamic properties. The results obtained in both one- and two-dimensional cases are expected to be interpreted in the context of energy distribution and optimization within the spin network [3].

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