On the closedness of the sum of marginal subspaces on $[a, b)$

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1. Let X be a real or complex Banach space. By a subspace of X we will mean a linear subset of X (thus a subspace of X *is not* necessarily closed in X). Let n be a natural number, $n \geqslant 2$, and let $X_1, ..., X_n$ be subspaces of X. Define their sum in the natural way, namely,

$$X_1 + ... + X_n := \{x_1 + ... + x_n \,|\, x_1 \in X_1, ..., x_n \in X_n\}.$$

It is clear that $X_1 + ... + X_n$ is a subspace of X. Assume that $X_1, ..., X_n$ are closed in X. The natural question arises: is $X_1 + ... + X_n$ closed in X? The question makes sense — the sum of two closed subspaces can be non-closed.

Systems of closed subspaces of Banach spaces for which the closedness of their sum is important arise in various branches of mathematics (see [1, Subsection 1.4]).

2. Let V be a vector space and $V_1, ..., V_n$ be subspaces of V. The subspaces $V_1, ..., V_n$ are said to be linearly independent if an equality $v_1 + ... + v_n = 0$, where $v_i \in V_i$ for $i = 1, 2, ..., n$, implies that $v_i = 0$ for $i = 1, 2, ..., n$.

3. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For an $\mathcal{F}$-measurable function (random variable) $\xi : \Omega \to \mathbb{R}$ denote by $E\xi$ the expectation of ξ (if it exists). Two random variables ξ and η are said to be equivalent if $\xi(\omega) = \eta(\omega)$ for μ-almost all ω. For $p \in [1, \infty)$ denote by $L^p(\mathcal{F}) = L^p(\Omega, \mathcal{F}, \mu)$ the set of equivalence classes of random variables $\xi : \Omega \to \mathbb{R}$ such that $E|\xi|^p < \infty$. For $\xi \in L^p(\mathcal{F})$ set $\|\xi\|_p = (E|\xi|^p)^{1/p}$. Then $L^p(\mathcal{F})$ is a Banach space. For every sub-σ-algebra $\mathcal{A}$ of $\mathcal{F}$ we define the marginal subspace corresponding to $\mathcal{A}$, $L^p(\mathcal{A})$, as follows. $L^p(\mathcal{A})$ consists of those elements (equivalence classes) of $L^p(\mathcal{F})$ which contain at least one $\mathcal{A}$-measurable random variable. The subspace $L^p(\mathcal{A})$ is closed in $L^p(\mathcal{F})$. Indeed, it is easily seen that $L^p(\mathcal{A})$ is canonically isometrically isomorphic to $L^p(\Omega, \mathcal{A}, \mu|_{\mathcal{A}})$. Since the latter space is Banach, we conclude that $(L^p(\mathcal{A}), \|\cdot\|_p)$ is complete. It follows that $L^p(\mathcal{A})$ is closed in $L^p(\mathcal{F})$. Denote by $L^p_0(\mathcal{A})$ the subspace of all $\xi \in L^p(\mathcal{A})$ with $E\xi = 0$. Clearly, this subspace is also closed in $L^p(\mathcal{F})$.

4. Our problem and result. Consider the space $\Omega = [a, b)$, where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, $a < b$. Denote by $\mathcal{B}([a, b))$ the Borel σ-algebra on $[a, b)$. Let μ be a probability measure on $\mathcal{B}([a, b))$ and $p \in [1, +\infty)$. For a sequence of points $\pi = \{a_1, a_2, a_3, a_4, ...\}$, where $a < a_1 < a_2 < a_3 < ...$ and $a_k \to b$ as $k \to \infty$, define the partition of $[a, b)$, $part(\pi)$, by

$$part(\pi) = \{[a, a_1), [a_1, a_2), [a_2, a_3), [a_3, a_4), ...\}.$$

Let $\sigma a(\pi)$ be the σ-algebra generated by $part(\pi)$. The marginal subspace $L^p(\sigma a(\pi))$ consists of those elements (equivalence classes) of $L^p(\mathcal{B}([a, b)))$ which contain at least one $\sigma a(\pi)$-measurable random variable, i.e., a random variable which is constant on each of the elements of $part(\pi)$. Note that if $\mu([a, a_1)) > 0$ and $\mu([a_k, a_{k+1})) > 0$ for every $k \geqslant 1$, then each equivalence class of random variables contains at most one $\sigma a(\pi)$-measurable random variable. Thus in this case every element of the marginal subspace $L^p(\sigma a(\pi))$ contains exactly one $\sigma a(\pi)$-measurable random variable.

We will study the following questions. Let $\pi_1, ..., \pi_n$ be sequences of points of $[a, b)$.

Question 1. *When is $L^p(\sigma a(\pi_1)) + ... + L^p(\sigma a(\pi_n))$ closed in $L^p(\mathcal{B}([a, b)))$?*

Question 2. *When are the subspaces $L_0^p(\sigma a(\pi_1)), ..., L_0^p(\sigma a(\pi_n))$ linearly independent?*

In the following theorem, proved in [2], we show that under mild conditions the marginal subspaces are linearly independent and establish a relation between closedness of the sum of marginal subspaces and "fast decreasing" of tails of the measure μ.

Theorem 1. *Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, $a < b$, μ be a probability measure on $\mathcal{B}([a, b))$, and $p \in [1, +\infty)$. Let $\pi_i = \{a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, ...\}$ be a sequence of points such that $a < a_1^{(i)} < a_2^{(i)} < a_3^{(i)} < ...$ and $a_k^{(i)} \to b$ as $k \to \infty$ for every $i = 1, 2, ..., n$. Assume that $\pi_i \cap \pi_j = \emptyset$, $i \neq j$. Let $\pi_1 \cup ... \cup \pi_n = \{b_2, b_3, b_4, ...\}$, where $a =: b_1 < b_2 < b_3 < b_4 < ...$. Assume that $\mu([b_k, b_{k+1})) > 0$ for every $k \geqslant 1$. The following statements are true.*

(1) The subspaces $L_0^p(\sigma a(\pi_1)), ..., L_0^p(\sigma a(\pi_n))$ are linearly independent.

(2) We have

$$\overline{L^p(\sigma a(\pi_1)) + ... + L^p(\sigma a(\pi_n))} = L^p(\sigma a(\pi_1 \cup ... \cup \pi_n))$$

and

$$\overline{L_0^p(\sigma a(\pi_1)) + ... + L_0^p(\sigma a(\pi_n))} = L_0^p(\sigma a(\pi_1 \cup ... \cup \pi_n)),$$

where $\overline{M}$ is the closure of the set M (in $L^p(\mathcal{B}([a, b)))$).

(3) If $\sup\{\mu([b_{k+1}, b))/\mu([b_k, b)) \mid k \geqslant 1\} < 1$, then

$$L^p(\sigma a(\pi_1)) + ... + L^p(\sigma a(\pi_n)) = L^p(\sigma a(\pi_1 \cup ... \cup \pi_n))$$

and

$$L_0^p(\sigma a(\pi_1)) + ... + L_0^p(\sigma a(\pi_n)) = L_0^p(\sigma a(\pi_1 \cup ... \cup \pi_n)).$$

(4) If the subspace $L^p(\sigma a(\pi_1)) + ... + L^p(\sigma a(\pi_n))$ is closed in $L^p(\mathcal{B}([a, b)))$ and there exists $k_0 \geqslant 1$ such that for arbitrary $i \in \{1, 2, ..., n\}$ and arbitrary $k \geqslant k_0$ the interval $(a_k^{(i)}, a_{k+1}^{(i)})$ contains a point $a_l^{(j)}$, then

$$\sup\{\mu([b_{k+1}, b))/\mu([b_k, b)) \mid k \geqslant 1\} < 1.$$

Corollary 1. *Assume that the conditions of Theorem 1 are satisfied. Assume that there exists $k_0 \geqslant 1$ such that for arbitrary $i \in \{1, 2, ..., n\}$ and arbitrary $k \geqslant k_0$ the interval $(a_k^{(i)}, a_{k+1}^{(i)})$ contains a point $a_l^{(j)}$. If the subspace $L^p(\sigma a(\pi_1)) + ... + L^p(\sigma a(\pi_n))$ is closed in $L^p(\mathcal{B}([a, b)))$ for some $p \in [1, +\infty)$, then the subspace $L^q(\sigma a(\pi_1)) + ... + L^q(\sigma a(\pi_n))$ is closed in $L^q(\mathcal{B}([a, b)))$ for arbitrary $q \in [1, +\infty)$.*

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