About the main relation of Wiman-Valiron theory and asymptotical h-density of an exeptional sets

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Let L be the class positive increasing on $[0;+\infty)$ functions, and L_1 be the subclass of the functions $h \in L$ such that $h\Big(x + \frac{1}{h(x)}\Big) = O(h(x)), (x \to +\infty)$. For a Lebesgue measurable set $E \subset [0;+\infty)$ of finite Lebesgue measure $\operatorname{meas} E = \int_E dx < +\infty$ we define its upper $D_h(E)$ and lower $d_h(E)$ asymptotic h-density on ∞ by the equalities

$$D_h(E) = \varlimsup_{R\to+\infty} h(R) \cdot \operatorname{meas}(E \cap [R;+\infty)), \quad d_h(E) = \varliminf_{R\to+\infty} h(R) \cdot \operatorname{meas}(E \cap [R;+\infty)).$$

Denote by $S(a,b)$, $-\infty \le a < b \le +\infty$, the class of analytic in $\Pi(a,b) = \{z : a < \operatorname{Re} z < b\}$ functions such that $(\forall x \in (a,b))\colon\ \ M(x,F) := \sup\{|F(t+iy)| : a < t \le x, y \in \mathbb{R}\} < +\infty$, where $L(x,F) = (\ln M(x,F))'_+$ is the right-hand derivative.

It is known (see [1, p.149, Theorem 1.3.17]) that for functions $F \in S(-\infty,+\infty)$ such that $L(x,F) \to +\infty$ $(x \to +\infty)$ the relation $F'(z) = (1+o(1))L(x,F)\cdot F(z)$ holds as $x \to +\infty$ outside some set E of finite Lebesgue measure, for all z such that $\operatorname{Re} z = x$ and $|F(z)| = (1+o(1))M(x,F)\quad x \to +\infty$.

This statement about the relation $F'(z) = (1+o(1))L(x,F)\cdot F(z)$ is used in the study of asymptotic properties of entire solution from the class $S(-\infty,+\infty)$ of differential equations (see, for example, [1]). Due to the existence of an exceptional set of finite logarithmic measure, the domain of applicability is limited to the class of an entire solutions of finite R-order $\varrho_R = \varlimsup_{x\to+\infty} \ln\ln M(x,F)/x < +\infty$. Therefore, there is a need for statements that would more precisely describe the value of an exceptional set. Such a statement is proved in [2].

Theorem 1. [2] *Let the functions $\Phi \in L$ and $h \in L_1$ be such that $h(r) = o(\Phi(r))$ $(r \to +\infty)$. If $F \in \mathcal{S}(0,+\infty)$ and $L(x,F) \ge \Phi(x)$ $(x \ge x_0)$, then the relation $F'(z) = (1+o(1))L(x,F)\cdot F(z)$ holds as $x \to +\infty$ $(x \notin E, \mathcal{D}_h(E) = 0)$ for each z, $\operatorname{Re} z = x$, such that the relation $|F(z)| = (1+o(1))M(x,F)$ holds as $x \to +\infty$.*

We replace the condition $L(x,F) \ge \Phi(x)$ in Theorem 1 with a much weaker condition. This is indicated by the following theorem.

Theorem 2. [3] *Let $\Phi, h \in L$ be the functions such that $h(2r) = o(\Phi(r))(r \to \infty)$. If $F \in S_\infty(0,\infty)$ and*

$$(\exists x_n \nearrow +\infty\ (n \to +\infty))\colon \quad L(x_n,F) \ge \Phi(x_n)\ (n \ge 1),$$

the relation $F'(z) = (1+o(1))L(x,F)F(z)$ holds as $x \to +\infty$ $(x \notin E,\ d_h(E) = 0)$ for all z such that $Rez = x$ and $|F(z)| = (1+o(1))M(x,F)$ as $x \to +\infty$.

However, Theorem 2, instead of the condition $h(r) = o(\Phi(r))$ $(r \to +\infty)$, contains the following stronger condition $h(2r) = o(\Phi(r))$ $(r \to +\infty)$.

Conjecture 1. In Teorem 2, the condition $h(2r) = o(\Phi(r))$ $(r \to \infty)$ can be replaced by the condition $h(r) = o(\Phi(r))$ $(r \to \infty)$.

References

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