Represenations and cohomologies of the alternating group of degree 4

Yu. Drozd[1], **A. Plakosh**[2]

[1]Harvard University, Cambridge, MA, USA

[1,2]Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

y.a.drozd@gmail.com, andrianaplakosh@gmail.com

Let A_4 be the alternating group of degree 4. We consider the *integral representations* of this group, that is $\mathbb{Z}\mathrm{A}_4$-modules M such that the abelian group of M is free of finite rank (A_4-*lattices*). Recall that a classification of 2-adic representations of A_4 was obtained by Nazarova [3]. Unfortunately, this classification gives no idea how to use it to calculate cohomologies of A_4-lattices. We propose another approach based on the technique of *Bäckström orders* [4]. Namely, since the group ring is always Gorenstein, all its 2-adic representations, except projective ones, are actually representations of an overring A [2]. In the case of $\mathbb{Z}_2\mathrm{A}_4$ this overring is a Bäckström order with the enveloping hereditary order $\mathbb{Z}_2 \times \mathbb{Z}_2[\theta] \times \mathrm{Mat}(3, \mathbb{Z}_2)$, where $\theta = \sqrt[3]{1}$, and the quotient $A/\mathrm{rad}A = \mathbb{F}_2 \times \mathbb{F}_4$. Using it, we relate 2-adic representations of A_4 with representations of the valued graph of type $\tilde{F}_4$ [1]:

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \xleftarrow{\;2,1\;} \circ \longrightarrow \circ,$$

where the fields associated with $\bullet$ are $\mathbb{F}_2$ and those associated with $\circ$ are $\mathbb{F}_4$. It allows to give a complete description of the Auslander-Reiten quiver of the category of A-lattices. We also describe all indecomposable integral representations of A_4 and explain non-uniqueness of decomposition of representations into indecomposables.

It is known that $\tau M \simeq \Omega M$ for every A-lattice M, where τ is the Auslander-Reiten transform and Ω is the syzygy of $\mathbb{Z}\mathrm{A}_4$-lattices [2]. Using it, we calculate Tate cohomologies of all A_2-lattices.

1. Dlab V., Ringel C. M. Indecomposable representations of graphs and algebras. Mem. Amer. Math. Soc., 1976, 73, 1–57.
2. Drozd Yu. A. Rejection lemma and almost split sequences. Ukr. Mat. Zh., 2021, 73, 908–929.
3. Nazarova L. A. Unimodular representations of the alternating group of degree four. Ukr. Mat. Zh., 1963, 15, 437–444.
4. Ringel C. M., Roggenkamp K. W. Diagrammatic methods in the representation theory of orders. J. Algebra, 1979, 60, 11–42.