

SWITCHING PROBLEM ON A GRAPH

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A. Skorokhod in 1961 formulated the reflection problem in [1], which subsequently proved to be an effective method for constructing reflected diffusions. Based on this reflection problem, a mapping was built that assigns to a function the solution of Skorokhod's reflection problem for that function. This mapping turned out to be useful for solving various problems in queueing theory (see, e.g. [2]). Skorokhod's reflection problem has been generalized by different authors. In particular, A. Pilipenko in [3] posed the problem of jump-like reflection, and this work made it possible to construct Brownian motion with jump-like reflection [4].

The aim of our work is to construct a similar mapping for natural gluing of functions on a graph. We investigate the properties of this mapping and apply it to the study of random walks, using the continuous mapping theorem and Donsker's invariance principle.

Definition 1. Let $m \in \mathbb{N}$. We say that the function \vec{X} is a solution to the *graph switching problem* for function \vec{W} and compensating function \vec{H} if for each $i \in \overline{1, m}$ the function $X_i(t)$ from the vector $\vec{X}(t) = (X_1(t), X_2(t), \dots, X_m(t))$ can be represented as

$$X_i(t) = W_i(T_i(t)) + H_i(\tilde{T}(t))$$

and there is a sequence $\{a_k, b_k \mid k \geq 0\} \subset \mathbb{R}_+$ such that this sequence and functions $T_i: \mathbb{R}_+ \mapsto \mathbb{R}_+$, $\tilde{T}: \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfy the following conditions:

1. $0 = a_0 \leq b_0 \leq a_1 \leq b_1 \leq \dots \leq a_n \leq b_n \leq \dots \leq +\infty$; $a_k \rightarrow +\infty$.
2. \tilde{T} is a non-decreasing right-continuous function and $\tilde{T}(0) = 0$.
3. For all $i \in \overline{1, m}$ the function T_i is a piecewise linear function that has derivative at each point of \mathbb{R}_+ except maybe $\{a_k, b_k \mid k \geq 0\}$.
4. For all $t \in \mathbb{R}_+ \setminus \{a_k, b_k \mid k \geq 0\}$ and for all $i \in \overline{1, m}$ we have $T'_i(t) \in \{0, 1\}$ and $T_i(0) = 0$.
5. For all $t \in A = \cup_{k=0}^{\infty} (a_k, b_k)$ holds $\sum_{i=1}^m T'_i(t) = 1$.
6. $\int_0^{\infty} \mathbb{I}_A(t) d\tilde{T}(t) = 0$.
7. For $B = \cup_{k=0}^{\infty} (b_k, a_{k+1})$ for all $i \in \overline{1, m}$ holds $\int_0^{\infty} \mathbb{I}_B(t) dT_i(t) = 0$.

We denote that \vec{X} is a solution of the graph switching problem for functions \vec{W} , \vec{H} in the following way: $\vec{X} \in \mathcal{R}(\vec{W}, \vec{H})$.

Theorem 1. For $n \in \mathbb{N}$ let $\vec{X}^{(n)}$ be a function from $\mathcal{R}(\vec{W}^{(n)}, \vec{H}^{(n)})$ and let the following conditions hold:

1. The sequence of functions $\{\vec{W}^{(n)}\}_{n \geq 1}$ converges locally uniformly to the function \vec{W} from $C(\mathbb{R}_+, \mathbb{R}^m)$ on \mathbb{R}_+ as $n \rightarrow \infty$.
2. The sequence of functions $\{\vec{H}^{(n)}\}_{n \geq 1}$ converges locally uniformly to the function \vec{H} from $C(\mathbb{R}_+, \mathbb{R}^m)$ on \mathbb{R}_+ as $n \rightarrow \infty$.
3. For all $T > 0$ and $\forall i \in \overline{1, m}$ we have $\liminf_{n \rightarrow \infty} \inf_{t \in [0, T]} X_i^{(n)}(t) \geq 0$.
4. For all $T > 0$ and $\forall i \in \overline{1, m}$ we have $\limsup_{n \rightarrow \infty} \sup_{t \in [0, T] \cap \overline{B^{(n)}}} X_i^{(n)}(t) \leq 0$.
5. The functions $H^{(n)}(t) = \sum_{i=1}^m H_i^{(n)}(t)$ converge locally uniformly to the function t on \mathbb{R}_+ as $n \rightarrow \infty$.
6. The sequence of functions $T^{(n)}(t) = \sum_{i=1}^m T_i^{(n)}(t)$ converges locally uniformly to the function t on \mathbb{R}_+ as $n \rightarrow \infty$.

Then there exists a subsequence n_k and functions $T_i: \mathbb{R}_+ \mapsto \mathbb{R}$, $i \in \overline{1, m}$ and $\tilde{T}: \mathbb{R}_+ \mapsto \mathbb{R}$ such that for all $i \in \overline{1, m}$ the functions $X_i^{(n_k)}$ converge locally uniformly to the function $X_i(\cdot) = W_i(T_i(\cdot)) + H_i(\tilde{T}(\cdot))$ as $k \rightarrow \infty$ and for all $i \in \overline{1, m}$ the functions $T_i^{(n_k)}$ converge to T_i and the functions $\tilde{T}^{(n_k)}$ converge to \tilde{T} locally uniformly. Moreover,

$$\begin{cases} \sum_{i=1}^m T_i(t) = t, \\ l_i(T_i) = H_i(\tilde{T}(t)), \quad i = \overline{1, m}. \end{cases}$$

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