On interpolation of distribution spaces associated with elliptic operators

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We study interpolation properties of complex distribution spaces $X(A,Y)$ given over a bounded Lipschitz domain Ω and associated with an elliptic differential operator A with C^∞-coefficients on $\overline{\Omega}$. If X and Y are quasi-Banach distribution spaces over Ω, then the space $X(A,Y)$ consists of all distributions $u\in X$ such that $Au\in Y$ and is endowed with the graph quasi-norm $\|u,X\|+\|Au,Y\|$. We suppose that X is either the Besov space $B^{s}_{p,q}(\Omega)$ or Triebel–Lizorkin space $F^{s}_{p,q}(\Omega)$ with $s\in\mathbb{R}$, $0<p\leq\infty$ ($p\neq\infty$ for F-spaces), and $0<q\leq\infty$. Let $\mathcal{U}(\Omega)$ stand for the class of all these spaces. They are Banach iff $p\geq1$ and $q\geq1$. If $\dim\Omega=2$, we suppose in addition that A is properly elliptic. Let 2ℓ be the even order of A.

We find sufficient conditions for Y under which the interpolation between the spaces $X(A,Y)$ preserves their structure. Let $\mathfrak{F}[\cdot,\cdot]$ be an arbitrary interpolation functor defined on the category of all interpolation pairs of quasi-Banach spaces (resp., Banach spaces). We say that a quasi-Banach space $Y\hookrightarrow\mathcal{D}'(\Omega)$ is admissible if a certain space from $\mathcal{U}(\Omega)$ is continuously embedded in Y. As usual, $\mathcal{D}'(\Omega)$ is the linear topological space of all distributions on Ω.

Theorem 1. *Let X_0 and X_1 be quasi-Banach (resp., Banach) spaces from the class $\mathcal{U}(\Omega)$, and let Y_0 and Y_1 be admissible quasi-Banach (resp., Banach) spaces embedded continuously in $\mathcal{D}'(\Omega)$. Given $j\in\{0,1\}$, we put $Z_j:=B^{s-2\ell}_{p,q}(\Omega)$ if $X_j:=B^{s}_{p,q}(\Omega)$ and put $Z_j:=F^{s-2\ell}_{p,q}(\Omega)$ if $X_j:=F^{s}_{p,q}(\Omega)$ (of course, s, p, and q depends on j). Then, up to equivalence of quasi-norms,*

$$\mathfrak{F}[X_0(A,Y_0),X_1(A,Y_1)]=(\mathfrak{F}[X_0,X_1])(A,\mathfrak{F}[Y_0\cap Z_0,Y_1\cap Z_1]).$$

Among applications of Theorem 1 to various interpolation functors, we focus our attention on $\pm$-interpolation functor $\mathfrak{F}[X_0,X_1]:=\langle X_0,X_1,\theta\rangle$ with the parameter $\theta\in(0,1)$. This functor is applicable to every interpolation pair of quasi-Banach spaces and retains the class of Triebel–Lizorkin spaces (in contrast to the real interpolation).

Theorem 2. *Let $E,G\in\{B,F\}$, and let $s_j,\alpha_j\in\mathbb{R}$ and $p_j,q_j,\beta_j,\gamma_j\in(0,\infty]$ for each $j\in\{0,1\}$. Suppose that $p_0<\infty$ and $p_1<\infty$ if $E=F$, and suppose that $\beta_0<\infty$ and $\beta_1<\infty$ if $G=F$. Assume also that $G^{\alpha_j}_{\beta_j,\gamma_j}(\Omega)\subset E^{s_j-2\ell}_{p_j,q_j}(\Omega)$ whenever $j\in\{0,1\}$. Define the parameters s, α, p, q, β, and γ by formulas $s:=(1-\theta)s_0+\theta s_1$, $\alpha:=(1-\theta)\alpha_0+\theta\alpha_1$, and*

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1},\quad \frac{1}{\beta}=\frac{1-\theta}{\beta_0}+\frac{\theta}{\beta_1},\quad \frac{1}{\gamma}=\frac{1-\theta}{\gamma_0}+\frac{\theta}{\gamma_1}.$$

Then, up to equivalence of quasi-norms,

$$\langle E^{s_0}_{p_0,q_0}(A,G^{\alpha_0}_{\beta_0,\gamma_0},\Omega),E^{s_1}_{p_1,q_1}(A,G^{\alpha_1}_{\beta_1,\gamma_1},\Omega),\theta\rangle=E^{s}_{p,q}(A,G^{\alpha}_{\beta,\gamma},\Omega).$$

Here, $E(A,G,\Omega)$ means the above-indicated space of all $u\in E(\Omega)$ such that $Au\in G(\Omega)$, where $E(\Omega)$ and $G(\Omega)$ symbolize Besov or Triebel–Lizorkin spaces over Ω.

These results have applications to elliptic boundary-value problems with rough data and are obtained together with A. A. Murach [1].

1. Chepurukhina I., Murach A. Distribution spaces associated with elliptic operators. arXiv preprint, 2024, arxiv:2406.08150, 42 p. https://arxiv.org/pdf/2406.08150