Laws of the iterated logarithm for iterated perturbed random walks

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Let $(\xi_k, \eta_k)_{k\geq 1}$ be independent identically distributed random vectors with arbitrarily dependent positive components and $T_k := \xi_1 + \ldots + \xi_{k-1} + \eta_k$ for $k \in \mathbb{N}$. We call the random sequence $(T_k)_{k\geq 1}$ a (globally) perturbed random walk. Consider a general branching process generated by $(T_k)_{k\geq 1}$ and let $Y_j(t)$ denote the number of the jth generation individuals with birth times $\leq t$. Assuming that $\operatorname{Var}\xi_1 \in (0,\infty)$ and allowing the distribution of η_1 to be arbitrary, we prove a law of the iterated logarithm (LIL) for $Y_j(t)$. For a family (x_t) of real numbers denote by $C((x_t))$ the set of its limit points.

Theorem 1. *Assume that $\sigma^2 := \operatorname{Var}\xi \in (0,\infty)$. Then, for $j \geq 2$,*

$$C\left(\left(\frac{Y_j(t) - \mathbb{E}Y_j(t)}{(2((2j-1)(j-1)!)^{-1}\sigma^2\mu^{-2j-1}t^{2j-1}\log\log t)^{1/2}} : t > \mathrm{e}\right)\right) = [-1, 1] \quad \text{a.s.},$$

where $\mu := \mathbb{E}\xi < \infty$.

This result complements the recently proved LIL for iterated standard random walk in [2]. In particular, we obtain a LIL for the counting process of $(T_k)_{k\geq 1}$.

Theorem 2. *Assume that $\sigma^2 = \operatorname{Var}\xi \in (0,\infty)$. Then*

$$C\left(\left(\frac{Y(t) - \mu^{-1}\int_0^t \mathbb{P}\{\eta \leq y\}\mathrm{d}y}{(2\sigma^2\mu^{-3}t\log\log t)^{1/2}} : t > \mathrm{e}\right)\right) = [-1, 1] \quad \text{a.s.},$$

where $\mu = \mathbb{E}\xi < \infty$.

The latter result was previously established in the article [1] under the additional assumption that $\mathbb{E}\eta_1^a < \infty$ for some $a > 0$. We show that the aforementioned additional assumption is not needed.

1. Iksanov A., Jedidi W., Bouzeffour F. A law of the iterated logarithm for the number of occupied boxes in the Bernoulli sieve. Statist. Probab. Letters., 2017, 126, 244–252.
2. Iksanov A., Kabluchko Z., Kotelnikova V. A law of the iterated logarithm for iterated random walks, with application to random recursive trees. ALEA, Lat. Am. J. Probab. Math. Stat., 2025, 22, 169–181.