Linear Boundary Value Problems for Systems of Integro-Differential Equations as Mathematical Models of Socio-Economic Processes

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In researching the solvability of various types of functional differential equations and boundary value problems for them, the theory of generalized inverse operators [1] has been widely used in the last decade. This approach allows, taking into account the specifics of each specific problem, to apply all the advantages of the "operator theory" for its solution. The specificity of studying the solvability and construction of solutions of IDE is that their linear part is an operator that does not have an inverse. The existence of solutions to such problems is studied within the framework of the theory of pseudoinverse matrices and operators, originally developed by A. M. Samoilenko and O. A. Boichuk and subsequently extended by their scientific school [3,5,7–9].

The study focuses on linear BVP for systems of integro-differential equations, with particular attention given to establishing conditions for their solvability.

Specifically, we consider the inhomogeneous weakly perturbed BVP for integrodifferential systems with impulsive effects at fixed moments of time

$$\dot{x}(t) - \Phi(t)\int\limits_a^b \Big[A(s)x(s) + B(s)\dot{x}(s)\Big]ds = f(t) + \varepsilon\int\limits_a^b \Big[K(t,s)x(s) + K_1(t,s)\dot{x}(s)\Big]ds, \quad t \neq \tau_i,$$
$$\Delta E_i x\mid_{t=\tau_i} := S_i x(\tau_i - 0) + \gamma_i + \varepsilon A_{1i} x(\tau_i - 0), \quad i = 1, ..., p,$$

with boundary value conditions

$$\ell x(\cdot,\varepsilon) = \alpha + \varepsilon \ell_1 x(\cdot,\varepsilon) \ \in \ \mathbb{R}^q.$$

Here, we use the assumptions and notation from [2,3,6,7]: $A(t)$, $B(t)$, $\Phi(t)$, $K(t,s), K_1(t,s)$ are, respectively, $m\times n$, $m\times n$, $n\times m$, $n\times n$, $n\times n$ matrices which components are sought in the space $L_2[a,b]$; column vectors of matrice $\Phi(t)$ are linearly independent at $[a,b]$; the $n\times 1$ vector function $f(t) \in L_2[a,b]$; E_i, S_i, A_{1i} are $k_i\times n$ constant matrices such that $\mathrm{rank}(E_i{+}S_i) = k_i < n$, which means that the corresponding components of solutions of the impulsive system admit unambiguous continuation through the points of discontinuity $\Delta E_i x\Big|_{t=\tau_i} := E_i(x(\tau_i{+}0) - x(\tau_i - 0))$; $\gamma_i \in \mathbb{R}^{k_i}$, $\ell = col(\ell_1, \ell_2, ..., \ell_q)$ is a bounded linear q-dimensional vector functional, $\alpha = \mathrm{col}(\alpha_1, \alpha_2, ..., \alpha_q) \in \mathbb{R}^q$.

The study of linear BVP for SIDE in the dynamic case offers a wide range of opportunities both for theoretical advancement and for practical applications. For example, impulsive SIDE with control mechanisms [3] can be employed to model and analyze various economic phenomena. One such example is the modeling of economic policy interventions in a macroeconomic context. In turn, integro-dynamic systems and their associated boundary value problems in appropriate functional spaces [4,5] serve as powerful tools for modeling processes with complex dynamics that take into account the system's memory — that is, physical, biological, or economic processes in which the current state depends on all previous states.

The breadth and applicability of this theory strongly motivate its further development.

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