On the h-measure of an exceptional set in Fenton-type theorem for Taylor-Dirichlet series

A. Yu. Bodnarchuk[1], **O. B. Skaskiv**[2]

[1]Ivan Franko National University of Lviv, Lviv, Ukraine

[2]Ivan Franko National University of Lviv, Lviv, Ukraine

bodnarchukmail@gmail.com, olskask@gmail.com

We consider the class $S(\lambda, \beta, \tau)$ of convergent for all $x \geq 0$ Taylor-Dirichlet type series of the form

$$F(x) = \sum_{n=0}^{+\infty} a_n e^{x\lambda_n + \tau(x)\beta_n},$$

where $a_n \geq 0$ for all $n \geq 0$, $\tau : [0, +\infty) \to (0, +\infty)$ is continuously differentiable non-decreasing function, $\lambda = (\lambda_n)$ and $\beta = (\beta_n)$ are such that $\lambda_n \geq 0$, $\beta_n \geq 0$ for all $n \geq 0$.

At International conference "Complex Analysis and Related Topics" (Lviv, September 23-28, 2013) ([1]) the following conjecture was formulated.

Conjecture 1 ([1]). The following statement is correct: For every sequences λ and β, functions τ, h, $\frac{h(x)}{x} \to +\infty$ $(x \to +\infty)$, there exist a function $F \in S(\lambda, \beta, \tau)$, a set E and a constant $d > 0$ such that $h - \text{meas}\ E := \int\limits_E dh(x) = +\infty$ and $\forall x \in E$ the inequality $F(x) > (1 + d)\mu(x, F)$ holds.

We give a partial answer to a question formulated in Conjecture 1.

Theorem 1 ([2]). For each increasing function $h(x) : [0, +\infty) \to (0, +\infty)$, $h'(x) \nearrow +\infty$ $(x \to +\infty)$, every sequence $\lambda = (\lambda_n)$ such that

$$\sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < +\infty$$

and for any non-decreasing sequence $\beta = (\beta_n)$ such that $\beta_{n+1} - \beta_n \leq \lambda_{n+1} - \lambda_n$ $(n \geq 0)$ there exist a function $\tau(x)$ such that $\tau'(x) \geq 1$ $(x \geq x_0)$, a function $F \in S(\lambda, \beta, \tau)$, a set E and a constant $d > 0$ such that h-meas $E := \int_E dh(x) = +\infty$ and

$$(\forall x \in E) : \ F(x) > (1 + d)\mu(x, F),$$

where $\mu(x, F) = \max\{|a_n|e^{x\lambda_n + \tau(x)\beta_n} : n \geq 0\}$ is the maximal term of the series.

Given Theorem 1, the following questions arise.

Question 1. Is the statement of Conjecture 1 correct in its entirety?

Question 2 ([1]). Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that $\frac{h(x)}{x} \to +\infty$, $(x \to +\infty)$. What are necessary and sufficient conditions that relationship

$$F(x) = (1 + o(1))\mu(x, F)$$

holds for $x \to +\infty$ $(x \notin E,\ h\text{-}meas E < +\infty)$ for every function $F \in S(\lambda, \beta, \tau)$?

The statement of Theorem 1 for the class $S(\lambda) := S(\lambda, 0, 0)$, that is, for the entire Dirichlet series, was proved earlier in paper [3].

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