Entire slice regular functions having bounded index of quaternionic variable

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We will use notations from [1] andc [2]. Let $\mathbb{H}$ be the skew field of quaternions which is defined as $\mathbb{H} = \{g = x_0 + ix_1 + jx_2 + kx_3 \colon x_0, x_1, x_2, x_3 \in \mathbb{R}\}$, where the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. It is a noncommutative field. We define the Euclidean norm on $\mathbb{H} : |q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$, real part is $Re\, q = x_0$, imaginary part is $Im\, q = ix_1 + jx_2 + kx_3$. The symbol $\mathbb{S}$ denotes the unit sphere of purely imaginary quaternions, i.e., $\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 \colon x_1^2 + x_2^2 + x_3^2 = 1\}$. One should observe that if $I \in \mathbb{S}$, then $I^2 = -1$. Given this, the elements of $\mathbb{S}$ are also called imaginary units. For any fixed $I \in \mathbb{S}$ we define $\mathbb{C}_I := \{x + Iy : x, y \in \mathbb{R}\}$. It is easy to check that $\mathbb{C}_I$ can be identified with a complex plane, moreover $\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I$. The real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real quaternion can be associated with any imaginary unit I. Any nonreal quaternion $q = x_0 + ix_1 + jx_2 + kx_3$ is uniquely associated to the element $I_q \in \mathbb{S}$ defined by

$$I_q := \frac{ix_1 + jx_2 + kx_3}{|ix_1 + jx_2 + kx_3|}.$$

It is obvious that q belongs to the complex plane $\mathbb{C}_{I_q}$.

Let $f : \mathbb{H} \to \mathbb{H}$ be real differentiable. The function f is said to be (left) entire slice regular or (left) entire slice hyperholomorphic if for every $I \in \mathbb{S}$, its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ passing through origin and containing I and 1 satisfies $\bar{\partial}_I f(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0$ on $\mathbb{C}_I$. The class of (left) slice regular functions on $\mathbb{H}$ will be denoted by $\mathcal{R}(\mathbb{H})$. Analogously, a function f is said to be right entire slice regular in $\mathbb{H}$ if $(f_I\bar{\partial}_I)(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I\right) = 0$ on $\mathbb{C}_I$. Let $f \in \mathcal{R}(\mathbb{H})$. The so-called left I-derivative of f at a point $q = x + Iy$ is defined by $\partial_I f(x + iy) := \frac{1}{2}(\frac{\partial}{\partial x} f_I(x + Iy) - I\frac{\partial}{\partial y} f_I(x + Iy))$ and the right I-derivative of f at $q = x+Iy$ is defined by $\partial_I f(x+iy) := \frac{1}{2}(\frac{\partial}{\partial x} f_I(x+Iy) - \frac{\partial}{\partial y} f_I(x+Iy)I)$. Let us now introduce another suitable notion of derivative. The slice derivative $\partial_s f$ of f, is defined by:

$$\partial_s(f)(q) = \begin{cases} \partial_I(f)(q), \text{ if } q = x + Iy, y \neq 0, \\ \frac{\partial f}{\partial x}(x), \text{ if } q = x \in \mathbb{R}. \end{cases}$$

We will often write $f'(q)$ instead of $\partial_s f(q)$. The k-th derivative of $f \in \mathcal{R}(\mathbb{H})$ is defined recursively as $f^{(k)}(q) = (f^{(k-1)}(q))'$. It is important to note that if $f(q)$ is a slice regular function then also $f'(q)$ is a slice regular function.

Let I, $J \in \mathbb{S}$ be such that I and J are orthogonal vectors, so that I, J, $IJ = K$ is a basis of $\mathbb{H}$ and write the restriction $f_I(x + Iy) = f(x + Iy)$ of f to the complex plane $\mathbb{C}_I$ as $f = f_0 + If_1 + Jf_2 + Kf_3$, where $f_0, \ldots, f_3$ are $\mathbb{R}$-valued. In alternative, it can also be written as $f = F + GJ$, where $f_0 + If_1 = F$, and $f_2 + If_3 = G$ are C_I-valued.

A function $f \in \mathcal{R}(\mathbb{H})$ is called a function of *bounded index,* if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $q \in \mathbb{H}$ the following inequality is valid

$$\frac{|f^{(m)}(q)|}{m!} \leq \max\left\{\frac{|f^{(k)}(q)|}{k!} : 0 \leq k \leq m_0\right\}.$$

The least such integer m_0 is called the *index of the entire slice regular function f and is denoted by $N(f) = m_0$.*

Theorem 1. *A function $f \in \mathcal{R}(\mathbb{H})$ is of bounded index if and only if for every $\eta > 0$ there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for any $I \in \mathbb{S}$ and for every $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}_+$ (or, equivalently, for any $q = x_0 + Iy_0 \in \mathbb{H}$) there exists $k_0 = k_0(q) = k_0(x_0, y_0, I) \in \mathbb{Z}_+$ with $0 \leq k_0 \leq n_0$ and the following inequality holds*

$$\max\{|f_I^{(k_0)}(x+Iy)| : \sqrt{(x-x_0)^2+(y-y_0)^2} \leq \eta\} \leq P_1|f_I^{(k_0)}(x_0+Iy_0)|.$$

These results will be published soon in [2]. More partial results for the Fueter regular function with boundex index were obtained in [1].

1. Baksa V. P., Bandura A. I. On an attempt to introduce a notion of bounded index for the Fueter regular functions of the quaternionic variable. Matematychni Studii, 2025, 60, 2, 191–200. https://doi.org/10.30970/ms.60.2.191-200
2. Baksa V. P., Bandura A. I., Skaskiv O. B. Entire slice regular functions having bounded index of quaternionic variable. Ukrainian Mathematical Journal, 2025, (accepted, in print).