Ostrowski-type inequalities in abstract distance spaces

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Estimates for the deviation between the value of an operator Λ at a function f from some class $\mathfrak{M}$ and the value of Λ at some depending on f constant function from $\mathfrak{M}$ play an important role in approximation theory and numeric analysis. For example, estimates for the deviation of a value of a function $f \in \mathfrak{M}$ at some point from its mean value is of this kind. One of the first among such sharp estimates (where $\Lambda f = \frac{1}{2}\int_{-1}^{1} f(t)dt$) was obtained by Ostrowski [1]:

Theorem 1. *Let $f\colon [-1,1] \to \mathbb{R}$ be a differentiable function and let for all $t \in (-1,1)$, $|f'(t)| \le 1$. Then for all $x \in [-1,1]$ the following inequality holds*

$$\left| \frac{1}{2}\int\limits_{-1}^{1} f(t)dt - f(x) \right| \le \frac{1+x^2}{2}.$$

The inequality is sharp in the sense that for each fixed $x \in [-1,1]$, the upper bound $\frac{1+x^2}{2}$ cannot be reduced.

The notion of a distance (in particular, a metric) plays an important role in many branches of mathematics. A set M with a reflexive, antisymmetric and transitive relation $\le$ is called partially ordered. Let X be an arbitrary set and M be a partially ordered set that has a smallest element, which we denote by θ (i.e., $\theta \le m$ for any $m \in M$). A function $h_X\colon X \times X \to M$ is called an *M-distance* in X, if for arbitrary $x, y \in X$, $h_X(x,x) = \theta$ and $h_X(x,y) = h_X(y,x)$. The pair (X, h_X) will be called an *M-distance space*. Speaking of a partially ordered set M, we assume that some M-distance h_M is defined in M. We say that an M-distance h_X in X *agrees* with an M-distance h_M in M, if

$$h_M(h_X(x,x_1), h_X(x,x_2)) \le h_X(x_1,x_2)\, \forall x, x_1, x_2 \in X.$$

Note that this inequality holds (and is equivalent to the triangle inequality) if $M = \mathbb{R}_+$ with the usual metric, and (X, h_X) is a pseudo metric space. An M-distance h_X on a set X will be called an *M-pseudo metric*, if it agrees with M-distance h_M. In this case the pair (X, h_X) will be called an *M-pseudo metric space*. We note that generally speaking an M-metric h_X need not agree with h_M. Moreover, an M-metric h_M does not necessarily agree with itself.

The class $H(X,Y)$ of mappings $f\colon X \to Y$ that satisfy the Lipschitz condition can be defined in a standard way, if distances h_X and h_Y are somehow defined in X and Y:

$$H(X,Y) = \{f\colon X \to Y\colon h_Y(f(x_1), f(x_2)) \le h_X(x_1,x_2)\ \ \forall x_1, x_2 \in X\}.$$

For an M-distance space X, an operator $\lambda\colon H(X,M) \to M$ will be called *monotone*, if for arbitrary $u, v \in H(X,M)$

$$(\forall x \in X\ u(x) \le v(x)) \implies (\lambda(u) \le \lambda(v)).$$

Let T, Y be M-distance spaces, X be an M-pseudo metric space, and $t \in T$ be fixed. We say that an operator $\Lambda\colon H(T,X) \to Y$ and a monotone operator $\lambda\colon H(T,M) \to M$ *agree*, if for all $f \in H(T,X)$

$$h_Y(\Lambda f(\cdot), \Lambda f(t)) \le \lambda(h_X(f(\cdot), f(t))).$$

Here and below $\Lambda f(t)$ means the value of the operator Λ on the constant function $\tau \mapsto f(t)$, $\tau \in T$ (the same notation will be used for other operators whose arguments are functions).

Our main result (see [2]) is the following Ostrowski-type inequality.

Theorem 2. *Let (T, h_T) and (X, h_X) be M-pseudo metric spaces, (Y, h_Y) be an M-distance space, and $t \in T$ be fixed. Assume that an operator $\Lambda\colon H(T,X) \to Y$ and a monotone operator $\lambda\colon H(T,M) \to M$ agree. Then for arbitrary function $f \in H(T,X)$ the following Ostrowski-type inequality holds:*

$$h_Y(\Lambda f(\cdot), \Lambda f(t)) \le \lambda(h_T(\cdot, t)). \tag{1}$$

If $\lambda(\theta) = \theta$, and there exists an operator $\phi_X\colon H(T,M) \to H(T,X)$ and $\phi_Y \in H(M,Y)$ with the property $h_Y(\phi_Y(m), \phi_Y(\theta)) = m$, if $m = \lambda(h_T(\cdot, t))$, such that the diagram

$$\begin{array}{ccc} H(T,X) & \xrightarrow{\Lambda} & Y \\ {\scriptstyle \phi_X}\uparrow & & \uparrow{\scriptstyle \phi_Y} \\ H(T,M) & \xrightarrow{\lambda} & M \end{array}$$

is commutative i.e., $\Lambda \circ \phi_X = \phi_Y \circ \lambda$, then inequality (1) is sharp and becomes equality on the function

$$f_t(\cdot) = \phi_X(h_T(\cdot, t)).$$

1. Ostrowski A. Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert. Comment. Math. Hel., 1938, 10, 226 – 227.
2. Babenko V. F., Babenko V. V., Kovalenko O. V. Ostrowski-type inequalities in abstract distance spaces. Res. Math., 2024, 32, No. 2, 9 – 20. doi:10.15421/242416.