Benjamin–Feir index for a two-layer fluid

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We examine the propagation of wave packets along the interface $z = \eta(x,t)$ between two incompressible fluid media, Ω_1 and Ω_2, with densities ρ_1 and ρ_2, respectively, taking into account the surface tension T acting on the interface $\eta(x,t)$. The regions in an undisturbed state have the following form: $\Omega_1 = \{(x,z) : |x| < +\infty, -h_1 < z < 0\}$ and $\Omega_2 = \{(x,z) : |x| < +\infty, 0 < z < h_2\}$ where the thickness of the layers are h_1 and h_2. The mathematical formulation in a dimensionless form of the wave packet propagation problem within this model takes the form

$$
\begin{gathered}
\Delta\phi_j = 0 \quad \text{in} \quad \Omega_j, \\
\eta_{,t} - \phi_{j,z} = -\alpha\eta_{,x}\phi_{j,x} \quad \text{when} \quad z = \alpha\eta(x,t), \\
\phi_{1,t} - \rho\phi_{2,t} + (1-\rho)\eta + 0.5\alpha\left(\nabla\phi_1\right)^2 - 0.5\alpha\rho\left(\nabla\phi_2\right)^2 - \\
-T\left(1 + (\alpha\eta_{,x})^2\right)^{-1.5}\eta_{,xx} = 0 \quad \text{when} \quad z = \alpha\eta(x,t), \\
\phi_{1,z} = 0 \quad \text{when} \quad z = -h_1, \quad \phi_{2,z} = 0 \quad \text{when} \quad z = h_2,
\end{gathered}
\tag{1}
$$

where $\alpha = a/l$ is a small dimensionless parameter representing the wave steepness, with a being the maximum displacement of the interface $\eta(x,t)$ and l the wavelength.

The surface elevation and velocity potentials in the domains $\Omega_j (j = 1,2)$ are represented according to the method of multiple scales

$$
(\eta, \phi_j) = \sum\nolimits_{n=1}^{3} \alpha^{n-1}(\eta_n, \phi_{jn}) + \mathrm{O}(\alpha^3), \tag{2}
$$

where $x_n = \alpha^n x$, $t_n = \alpha^n t$ are the spatial and temporal scaling variables, η_n and ϕ_{jn} - terms in the asymptotic expansion involving slow-scale variables. The substitution (2) into the problem (1) leads to the first three linear approximations with respect to the unknown functions, which are coefficients in the expansion (2). The solutions of the first- and second-order approximations of the investigated problem have been found. From the second- and third-order approximation problems, the solvability conditions were obtained, which, together with the dispersion relation $\omega^2 = k(1-\rho+Tk^2)(\coth kh_1 + \rho\coth kh_2)^{-1}$, lead to an evolution equation for the wave packet envelope A in the form of a nonlinear Schrödinger equation

$$
iA_{,t} + i\omega' A_{,x} + 0.5\omega'' A_{,xx} = -\alpha\omega^{-1}JA^2\bar{A} \tag{3}
$$

where $A = A(x_1, x_2, t_1, t_2)$ is the envelope of the wave packet, $\overline{A}$ is the complex conjugate of A, k is the wave number, ω is the frequency of the wave packet center, $\theta = kx_0 - \omega t_0$, $\omega' = \partial\omega/\partial k$ is the group velocity, $\omega'' = \partial^2\omega/\partial k^2$, and the Benjamin–Feir index J is expressed in the form

$$
\begin{gathered}
J = -[16(1-\rho)(\rho\coth kh_2 + \coth kh_1)]^{-1}\{2k\omega^2(1-\rho)\Lambda[-3\rho\coth^2 kh_2 + \\
+3\coth^2 kh_1 - 1 + \rho] - 4k\omega^4[\rho(\coth^2 kh_2 - 1) - (\coth^2 kh_1 - 1)] - 4k^2\omega^2(1-\rho)[\rho\coth^3 kh_2 \\
+\coth^3 kh_1 - 2\rho\coth kh_2 - 2\coth kh_1] - 3Tk^5(1-\rho)\},
\end{gathered}
\tag{4}
$$

$$\Lambda = 0.5k\omega^2[\rho\coth^2 kh_2 - \coth^2 kh_1 + 4\rho\coth 2kh_2 \coth kh_2 - 4\coth 2kh_1 \coth kh_1 -$$
$$-3(\rho-1)][4Tk^3 - k\rho + k - 2\omega^2(\rho\coth 2kh_2 + \coth kh_1)]^{-1},$$

where Λ is an amplitude of the second harmonic in the expansion (2) of the interface $\eta(x,t)$.

Let us consider a solution of equation (3) that depends only on time $A = a\exp(i\alpha a^2\omega^{-1}Jt)$, where a is the amplitude of the envelope. Further, using the methodology described in [1], we obtain the modulational stability condition for the envelope in the form:

$$J\omega'' < 0. \tag{5}$$

Based on condition (5), the modulational stability of the envelope has been investigated. Figure 1 presents the modulational stability diagram (MSD) for $T=1$, $h_1=4$, $h_2=3$. The MSD is divided into regions of linear instability (dark shading) and linear stability. The region of linear stability, in turn, consists of areas of nonlinear stability (unshaded) and nonlinear instability or Benjamin-Feir instability (light shading). The region of linear stability is divided into regions of nonlinear stability and instability by curves along which $J=0$ (red curves), $J\to\infty$ (blue curves), and $\omega''=0$ (green curves). It should be noted that in the case of equal layer thicknesses $h_1=h_2=h$, the vertical asymptote at $\rho=1$ on MSD is absent. In this case, J has a finite limit at this point, which is equal to $32^{-1}Tk^5(10\sinh^{-1}2kh - \coth kh)$.

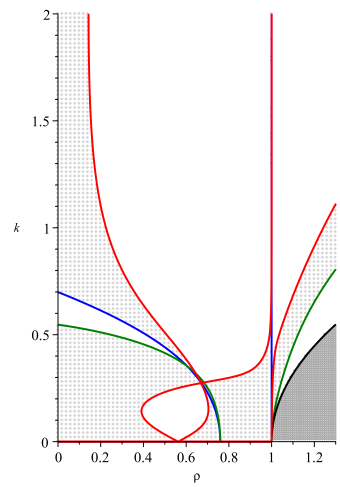

Figure 1: MSD for $h_1=4, h_2=3$

For model (1) limiting cases were analyzed in which the layer thicknesses are large compared to the wave length. The first case corresponds to the situation where the thickness of the lower layer tends to infinity. In this case, where $h_1\to-\infty$, the Benjamin–Feir index J and the amplitude of the second harmonic Λ are given by the expression

$$J = -[16(1-\rho)(\rho\coth kh_2+1)]^{-1}[2k\omega^2(1-\rho)\Lambda(-3\rho\coth^2 kh_2+\rho+2) - 4k\omega^4(\rho\coth^2 kh_2-\rho)^2$$
$$-4k^2\omega^2(1-\rho)(\rho(\coth^3 kh_2 - 2\rho\coth kh_2 - 1) - 3Tk^5(1-\rho)],$$
$$\Lambda = 0.5k\omega^2[\rho\coth kh_2(4\coth 2kh_2+\coth kh_2) - 3\rho - 2][4Tk^3 - k\rho + k - 2\omega^2(\rho\coth 2kh_2 + 1)]^{-1}.$$

In the second case, when $h_2\to\infty$, the Benjamin–Feir index J and amplitude Λ are given by

$$J = -[16(1-\rho)(\rho+\coth kh_1)]^{-1}[2k\omega^2(1-\rho)\Lambda(3\coth^2 kh_1 - 2\rho - 1) - 4k\omega^4(-\coth^2 kh_1 + 1)$$
$$-4k^2\omega^2(1-\rho)(\coth^3 kh_1 - 2\coth kh - \rho) - 3Tk^5(1-\rho)],$$
$$\Lambda = 0.5k\omega^2[3+2\rho - \coth kh_1(4\coth 2kh_1 + \coth kh_1)][4Tk^3 - k\rho + k - 2\omega^2(\coth 2kh_1 + \rho)]^{-1}.$$

When both layer thicknesses tend to infinity, the result converges to Nayfeh's earlier solution [1], confirming its validity.

1. Nayfeh A. Nonlinear propagation of wave-packets on fluid interface. Trans. ASME, Ser. E: J. Appl. Mech., 1976, 43, 3, 584-588.