Approximation problem for generic boundary-value problems in Sobolev spaces

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Let $(a, b) \subset \mathbb{R}$ be a finite interval and the parameters $n \in \mathbb{N} \cup \{0\}$, $\{m, r\} \subset \mathbb{N}, 1 \leq p \leq \infty$, $W_p^{n+r}\big([a, b]; \mathbb{C}\big)$ be a complex Sobolev space.

Let us consider a linear boundary-value problem

$$(L_0 y)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a, b), \tag{1}$$

$$B_0 y = c, \tag{2}$$

where the matrix-valued functions $A_{r-j}(\cdot) \in (W_p^n)^{m \times m}$, the vector-valued function $f(\cdot) \in (W_p^n)^m$, the vector $c \in \mathbb{C}^{rm}$ and the linear continuous operator

$$B_0 \colon (W_p^{n+r})^m \to \mathbb{C}^{rm}. \tag{3}$$

A solution to the boundary-value problem (1), (2) is understood as a vector-valued function $y_0(\cdot) \in (W_p^{n+r})^m$ that satisfies both equation (1) (everywhere if $n \geq 1$, and almost everywhere if $n = 0$) on (a, b) and equality (2). The boundary conditions may contain derivatives of the unknown functions of integer and fractional orders that exceed the order of the differential system.

We rewrite the problem (1), (2) in the form of a linear operator equation $\big(L_0, B_0\big) y_0 = c$. Here, $\big(L_0, B_0\big)$ is a bounded linear operator on the pair of Banach spaces

$$\big(L_0, B_0\big) \colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^{rm}. \tag{4}$$

According to [1, Theorem 1], the operator (3) is a Fredholm one with zero index.

Any linear continuous operator (4) for $1 \leq p < \infty$ admits a unique analytic representation

$$By = \sum_{j=0}^{n+r-1} \alpha_k \, y^{(k)}(a) + \int_a^b \Phi(t) y^{(n+r)}(t) \mathrm{d}t, \quad y(\cdot) \in (W_p^{n+r})^m.$$

for some numerical matrices $\alpha_k \in \mathbb{C}^{rm \times rm}$ and matrix-valued functions $\Phi(\cdot) \in L_{p'}([a, b]; \mathbb{C}^{rm \times rm})$; as usual, $1/p + 1/p' = 1$.

Let us consider a sequence of operators (L_k, B_k) of the form (1), (3) and investigate the question of their uniform convergence to a given operator (L_0, B_0).

Theorem 1. *In the assumptions made, the operators (L_k, B_k) uniform converge to the given operator (L_0, B_0) if and only if the following three conditions are fulfilled as $k \to \infty$:*

(I) $A_{r-j}(\cdot, k) \to A_{r-j}(\cdot)$ in the Banach space $(W_p^n)^{m \times m}$ for each $j \in \{1, \ldots, r\}$;

(II) $\alpha_s(k) \to \alpha_s$ y $\mathbb{C}^{rm \times rm}$ for each $s \in \{0, \ldots, n + r\}$;

(III) $\|\Phi(\cdot,k)-\Phi(\cdot)\|_{p'}\to 0$, *where* $\frac{1}{p}+\frac{1}{p'}=1$, $1\le p<\infty$.

Based on Theorem 1, it is possible to prove the following theorem about the approximation of the solutions of problem (1), (2) by a sequence of solutions of problems with polynomial coefficients and multipoint boundary conditions.

Theorem 2. *Let* $1<p<\infty$ *and the inhomogeneous boundary-value problem (1), (2) be well-defined. Then there exists a sequence of operators* L_k *with polynomial coefficients and boundary-value operators of the form*

$$B_k y=\sum_{j=0}^{n+r-1}\alpha_j(k)\,y^{(j)}(a)+\sum_{i=0}^{N}\beta_{i,k}y^{(n+r-1)}(t_{i,k}),$$

such that the operators (L_k,B_k) *are invertible and*

$$\left\|\,(L_k,B_k)^{-1}-(L_0,B_0)^{-1}\,\right\|\to 0\quad as\quad k\to\infty.$$

From Theorem 2 it follows that the solutions of the problems $L_k y_k=f$, $B_k y=c$ converge to the solution of the boundary-value problem (1), (2) in the space $(W_p^{n+r})^m$ and this convergence is uniform on bounded subsets of the right-hand sides of the inhomogeneous boundary-value problem (1), (2).

Remark 1. Note that instead of polynomial coefficients $A_{r-j}(\cdot,k)$ we can take an arbitrary dense subset of the space $(W_p^n)^m$.

Remark 2. The condition of $p>1$ in Theorem 2 is caused by the fact that the set of step functions is not a dense subset of a Banach space L_∞, and the condition of $p<\infty$ in this theorem is related to the use of formula (4).

1. Mikhailets V., Atlasiuk O. Differential systems in Sobolev spaces with generic inhomogeneous boundary conditions. Carpathian Math. Publ., 2024, 16, 2, 523–538.