Graphs with odd distances between cut and non-cut vertices

K. O. Antoshyna[1,2], S. O. Kozerenko[2]
[1]Institute of Mathematics NAS of Ukraine, Kyiv, Ukraine
[2]Kyiv School of Economics, Kyiv, Ukraine
kantoshyna@imath.kiev.ua, kozerenkosergiy@ukr.net

A vertex u in a finite connected graph G is called a cut vertex if its removal results in a disconnected graph $G - u$. A graph is biconnected provided it does not contain cut vertices. In [1], connected graphs with even distances between their non-cut vertices (NCE-graphs) and graphs with these odd distances (NCO-graphs) were considered.

Here we consider graphs with parity conditions on distances between pairs of vertices, where one is a cut vertex and the other is a non-cut vertex. At first, we observe that biconnected graphs are precisely the graphs in which the distances between cut and non-cut vertices are even. Indeed, if G has at least one cut vertex u, there would be a non-cut vertex v adjacent to u (as G is connected). The situation becomes more intriguing when considering graphs with these distances being odd.

A graph is called a CNCO-graph if it has odd distances between pairs of its vertices with different cutness. Clearly, biconnected graphs are also CNCO-graphs. A graph is called a CE-graph if it has even distances between cut vertices.

Theorem 1. *Let G be a connected, but not a biconnected graph. Consider the following three statements:*

1. *G is a CE-graph and an NCE-graph;*
2. *G is a CNCO-graph;*
3. *G is a CE-graph having dominating set of its cut vertices.*

Then the next implications hold: $1 \Rightarrow 2 \Rightarrow 3$.

If we restrict ourselves to the class of bipartite graphs, then the three statements from Theorem 1 become equivalent.

Corollary 1. *Let G be a connected, but not a biconnected bipartite graph. Then the following statements are equivalent:*

1. *G is a CE-graph and an NCE-graph;*
2. *G is a CNCO-graph;*
3. *G is a CE-graph having dominating set of its cut vertices.*

It is also easy to characterize CNCO-graphs with a unique cut vertex.

Corollary 2. *A graph G is a CNCO-graph with a unique cut vertex if and only if $G \simeq K_1 + H$ for a disconnected graph H (here $+$ denotes the graph join operation).*

A signed graph is a pair (G, σ), where G is a graph, and σ is a sign function that assigns one of two signs $\sigma(e) \in \{+, -\}$ to each edge e of the graph G (see [5]). Edges e with $\sigma(e) = +$ are called positive, and with $\sigma(e) = -$ are called negative.

For a graph G, it is natural to consider a sign function $\sigma_c : E(G) \to \{+, -\}$, where $\sigma_c(uv) = +$ if u and v have the same cutness (are both cut or non-cut vertices), and $\sigma_c(uv) = -$ provided u and v have different cutness. We refer to the value $\sigma_c(e)$ as to the cut-sign of an edge e. It is clear that the resulting signed graph is balanced (see [3]): each cycle in it has an even number of negative edges. In such a context, CNCO graphs can be characterized as follows.

Theorem 2. *Let G be a connected graph. Then G is a CNCO-graph if and only if every induced P_3 in G has the edges with the same cut-sign.*

It is also can be observed that in a CNCO-graph, the set of cut vertices must be independent. However, we can still unveil some of the structure of this set using the following construction. For a connected graph G, its 2-distance graph is the graph $D_2(G)$ having the same vertex set $V(D_2(G)) = V(G)$ and the edge set $E(D_2(G)) = \{uv : d_G(u, v) = 2\}$ (here $d_G(u, v)$ denotes the usual graph distance between u and v). We note that the paper [4] contains the more general definition of k-distance graphs, and the paper [2] presents a characterization of 2-distance graphs.

One can easily show that the set of cut vertices in a CNCO-graph G induces a connected subgraph in its 2-distance graph. Indeed, assume $u, v \in V(G)$ are two cut vertices in G. Then the distance $d_G(u, v)$ must be even. If $d_G(u, v) = 2$, then u and v are adjacent in $D_2(G)$. Otherwise, let $d_G(u, v) \geq 4$. Fix a shortest path $u - x_1 - x_2 - \cdots - v$. As $d_G(u, x_2) = 2$, the vertex x_2 is also a cut vertex. Thus, any two cut vertices from G are joined by a path in $D_2(G)$.

The next result shows that there are no other restrictions on the structure of the subgraph induced by the cut vertices of a CNCO-graph G in its 2-distance graph $D_2(G)$.

Theorem 3. *For any connected graph H, there exists a CNCO-graph G such that the subgraph of $D_2(G)$ induced by the cut vertices in G, is isomorphic to H.*

Acknowledgements

The authors are deeply grateful to Armed Forces of Ukraine for their unwavering daily heroism in safeguarding Ukraine. Kateryna Antoshyna was supported by a grant from the Simons Foundation (1030291, K.O.A.).

1. Antoshyna K., Kozerenko S. Graphs with odd and even distances between non-cut vertices. Opuscula Math., 2025, 45, No. 1, 5–25.
2. Ching R. P., Garces I. J. L. Characterizing 2-distance graphs. Asian-Eur. J. Math., 2019, 12, No. 1, 1950006 (10 pages).
3. Harary F. On the notion of balance of a signed graph. Michigan Math. J., 1953/1954, 2, No. 2, 143–146.
4. Harary F., Hoede C. and Kadlecek D. Graph-valued functions related to step graphs. J. Comb. Inf. Syst. Sci., 1982, 7, 231–246.
5. Zaslavsky T. Signed graphs. Discrete Appl. Math, 1982, 4, 47–74.