On Triebel–Lizorkin spaces of generalized smoothness on manifolds

A. V. Anop

Georg-August-Universität Göttingen, Mathematisches Institut, Göttingen, Germany;
Institute of Mathematics of the NAS of Ukraine, Kyiv, Ukraine
anna.anop@mathematik.uni-goettingen.de, anop@imath.kiev.ua

We consider a wide class of normed Triebel–Lizorkin spaces $F^{\alpha}_{p,q}$ of generalized smoothness α that admit the localization on compact C^{∞}-manifolds. This property follows from the fact that such spaces are obtained by double (in two steps) interpolation between corresponding spaces of classical smoothness given by a number. We give an application of the spaces $F^{\alpha}_{p,q}$ to elliptic pseudodifferential operators (PsDOs) on manifolds.

Let OR denote the class of all Borel functions $\alpha:[1,\infty)\to(0,\infty)$ for each of which there exist numbers $b>1$ and $c\geq 1$ such that $c^{-1}\leq\alpha(\lambda t)/\alpha(t)\leq c$ for every $t\geq 1$ and $\lambda\in[1,b]$ (b and c may depend on α). Such functions are called O-regularly varying at infinity and was introduced by V. G. Avakumović. This definition is equivalent to the following: there exist real numbers $r_0\leq r_1$ and positive numbers c_0 and c_1 such that $c_0\lambda^{r_0}\leq\alpha(\lambda t)/\alpha(t)\leq c_1\lambda^{r_1}$ for all $t,\lambda\geq 1$. Let $\sigma_0(\alpha)$ (resp., $\sigma_1(\alpha)$) denote the supremum of all r_0 (resp., the infimum of all r_1) such that the left-hand (resp., right-hand) inequality holds true for all $t,\lambda\geq 1$. The numbers $\sigma_0(\alpha)$ and $\sigma_1(\alpha)$ are called the Matuszewska indices of α.

Let $\alpha\in\mathrm{OR}$, $p,g\in(1,\infty)$, and $1\leq n\in\mathbb{Z}$. By definition, the complex linear space $F^{\alpha}_{p,q}(\mathbb{R}^n)$ consists of all tempered distributions w on $\mathbb{R}^n$ such that

$$\|w,F^{\alpha}_{p,q}(\mathbb{R}^n)\|^p=\int\limits_{\mathbb{R}^n}\biggl(\sum_{j=0}^{\infty}\alpha^q(2^j)\,|(\mathcal{F}^{-1}[\phi_j\mathcal{F}w])(x)|^q\biggr)^{p/q}dx<\infty.$$

This space is endowed with the norm $\|w,F^{\alpha}_{p,q}(\mathbb{R}^n)\|$. Here, the functions $0\leq\phi_j\in C^{\infty}(\mathbb{R}^n)$ form a resolution of unity on $\mathbb{R}^n$ that satisfy the following two conditions: $\operatorname{supp}\phi_0\subset\{\xi\in\mathbb{R}^n:|\xi|\leq 2\}$ and $\operatorname{supp}\phi_j\subset\{\xi\in\mathbb{R}^n:2^{j-1}\leq|\xi|\leq 2^{j+1}\}$ whenever $1\leq j\in\mathbb{Z}$, and, for any n-dimensional multi-index μ, there exists a positive number c_μ such that $|\xi|^{|\mu|}|\partial^{\mu}\phi_j(\xi)|\leq c_\mu$ whenever $\xi\in\mathbb{R}^n$ and $0\leq j\in\mathbb{Z}$. As usual, $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and its inverse.

The space $F^{\alpha}_{p,q}(\mathbb{R}^n)$ is complete and does not depend up to equivalence of norms on the above choice of functions ϕ_j. If $\alpha(t)\equiv t^s$ for a certain number $s\in\mathbb{R}$ (the classical smoothness), then $F^{\alpha}_{p,q}(\mathbb{R}^n)$ becomes the Triebel–Lizorkin space $F^{(s)}_{p,q}(\mathbb{R}^n)$ of order s.

The Hilbert space $F^{\alpha}_{2,2}(\mathbb{R}^n)=:H^{\alpha}(\mathbb{R}^n)$ form the extended Sobolev scale [1, Section 2.4.2]. This space is obtained by the quadratic interpolation (with an appropriate function parameter ψ) between Sobolev spaces. Namely,

$$H^{\alpha}(\mathbb{R}^n)=(H^{(s_0)}(\mathbb{R}^n),H^{(s_1)}(\mathbb{R}^n))_{\psi},$$

where the numbers s_0 and s_1 satisfy $s_0<\sigma_0(\alpha)$ and $s_1>\sigma_1(\alpha)$, whereas the interpolation function parameter ψ is defined by $\psi(t):=t^{-s_0/(s_1-s_0)}\alpha(t^{1/(s_1-s_0)})$ if $t\geq 1$ and by $\psi(t):=\alpha(1)$ if $0<t<1$. The Hilbert space $(E_0,E_1)_{\psi}$ is the result of the quadratic interpolation with the parameter ψ between Hilbert spaces E_0 and E_1 such that E_1 is continuously embedded in E_0.

Every space $F^{\alpha}_{p,q}(\mathbb{R}^n)$ is obtained by the complex interpolation between a certain space $H^{\alpha_0}(\mathbb{R}^n)$ with $\alpha_0\in\mathrm{OR}$ and a certain Triebel–Lizorkin space $F^{(s_1)}_{p_1,q_1}(\mathbb{R}^n)$ with $s_1\in\mathbb{R}$.

Theorem 1. *Let $\alpha \in \mathrm{OR}$ and $p,q \in (1,\infty)$. If $p \neq 2$, then we choose a number p_1 such that $1 < p_1 < p < 2$ or $2 < p < p_1 < \infty$, define a number $\theta \in (0,1)$ by the formula $1/p = (1-\theta)/2 + \theta/p_1$ and a number q_1 by $1/q = (1-\theta)/2 + \theta/q_1$, and assume that $q_1 \in (1,\infty)$ (specifically, this assumption holds if $|p-p_1|$ is sufficiently small). If $p = 2$ and $q \neq 2$, then we put $p_1 = 2$, choose a number q_1 such that $1 < q_1 < q < 2$ or $2 < q < q_1 < \infty$ and denote the number $\theta \in (0,1)$ by the formula $1/q = (1-\theta)/2 + \theta/q_1$. In both cases, we arbitrarily choose the number $s_1 \in \mathbb{R}$ and define a function $\alpha_0 \in \mathrm{OR}$ by the formula $\alpha_0(t) := (t^{-\theta s_1}\alpha(t))^{1/(1-\theta)}$ for every $t \geq 1$. Then, up to equivalence of norms,*

$$F^{\alpha}_{p,q}(\mathbb{R}^n) = [H^{\alpha_0}_2(\mathbb{R}^n), F^{(s_1)}_{p_1,q_1}(\mathbb{R}^n)]_\theta.$$

As usual, the Banach space $[E_0, E_1]_\theta$ is the result of the complex interpolation with the number parameter θ between Banach spaces E_0 and E_1 that are continuously embedded in a certain linear Hausdorff topological space.

Let M be a compact oriented boundaryless manifold of class C^∞ and dimension n. We arbitrarily choose a finite collection of local charts $\pi_j : \mathbb{R}^n \leftrightarrow U_j$ on M, with $j = 1, \ldots, \varkappa$. Here, $\{U_j\}$ is a covering of M by open sets. We also choose functions $\chi_j \in C^\infty(M)$, with $j = 1, \ldots, \varkappa$, such that $\chi_1 + \cdots + \chi_\varkappa = 1$ on M and that $\operatorname{supp}\chi_j \subset U_j$.

By definition, the complex linear space $F^{\alpha}_{p,q}(M)$ consists of all distributions f on M such that $(\chi_j f) \circ \pi_j \in F^{\alpha}_{p,q}(\mathbb{R}^n)$ for each $j \in \{1, \ldots, \varkappa\}$. Here, $(\chi_j f) \circ \pi_j$ is the image of the distribution $\chi_j f$ in the local chart π_j. This space is endowed with the norm

$$\|f, F^{\alpha}_{p,q}(M)\| := \|(\chi_1 f) \circ \pi_1, F^{\alpha}_{p,q}(\mathbb{R}^n)\| + \cdots + \|(\chi_\varkappa f) \circ \pi_\varkappa, F^{\alpha}_{p,q}(\mathbb{R}^n)\|.$$

Theorem 2. *The normed space $F^{\alpha}_{p,q}(M)$ is complete and does not depend (up to equivalence of norms) on the indicated choice of the atlas $\{\pi_j\}$ and partition of unity $\{\chi_j\}$ on the manifold M. Theorem 1 (and the above result for the quadratic interpolation) remain valid for the corresponding spaces on M.*

Applying this theorem to PsDOs acting between Triebel–Lizorkin spaces of classical smoothness, we obtain the following result:

Theorem 3. *Let A be an elliptic polyhomogeneous PsDO on M of order $r \in \mathbb{R}$. This operator is bounded and Fredholm on the pair of spaces $F^{\alpha}_{p,q}(M)$ and $F^{\alpha\varrho^{-r}}_{p,q}(M)$ whatever $\alpha \in \mathrm{OR}$ and $p, g \in (1,\infty)$. Its finite-dimensional kernel lies in $C^\infty(M)$ and together with its finite index does not depend on α, p, and q. Here, $\varrho(t) := t^r$, so that $\alpha\varrho^{-r}$ denotes the function $\alpha(t)t^r$.*

Analogous results hold true for the Besov space $B^{\alpha}_{p,q}$ of generalized smoothness $\alpha \in \mathrm{OR}$. However, as is known, this space is gotten by the real interpolation with a function parameter between Besov spaces of classical (number) smoothness.

These results are obtained together with A. A. Murach [2,3], specifically for L_p-Sobolev spaces of generalized smoothness.

1. Mikhailets V. A, Murach A. A. Hörmander Spaces, Interpolation, and Elliptic Problems. — Berlin: De Gruyter, 2014.
2. Anop A., Murach A., Interpolation spaces of generalized smoothness and their applications to elliptic equations (Ukrainian). arxiv:2110.06050.
3. Anop A., Murach A., Elliptic boundary-value problems in some distribution spaces of generalized smoothness. arxiv:2503.20747.