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Characterization of invertible operators in $\delta(\mathcal{H})$ via Duggal transform

S. Zid[1], **S. Menkad**[2]

[1]Department of mathematics, Faculty of Mathematics and Informatics , University of Batna 2, Batna, Algeria

[2]Department of mathematics, Faculty of Mathematics and Informatics , University of Batna 2, Batna, Algeria

s.zid@univ-batna2.dz, s.Menkad@univ-batna2.dz

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{N}(T)$ and T^* for the null subspace and the adjoint operator of T, respectively.

Recall that for $T \in \mathcal{B}(\mathcal{H})$, there is a unique factorization $T = U|T|$, where $\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T. This factorization is called the polar decomposition of T. It is known that if T is invertible then U is unitary and $|T|$ is also invertible. From the polar decomposition, the Aluthge transform of T is defined by

$$\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}, \quad T \in \mathcal{B}(\mathcal{H}).$$

This transform was introduced in [1] by Aluthge, in order to study p-hyponormal. In [2], Okubo introduced a more general notion called λ-Aluthge transform which has later been studied also in detail. This is defined for any $\lambda \in [0, 1]$ by

$$\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}, \quad T \in \mathcal{B}(\mathcal{H}).$$

Clearly, for $\lambda = \frac{1}{2}$ we obtain the usual Aluthge transform. Also, $\Delta_1(T) = |T|U$ is known as Duggal's transform. These transforms have been studied in many different contexts and considered by a number of authors. One of the interests of the Aluthge transform lies in the fact that it respects many properties of the original operator. For example T has a nontrivial invariant subspace if an only if $\Delta(T)$ does. Another important property is that T and $\Delta_\lambda(T)$ have the same spectrum. So T is invertible if and only if $\Delta_\lambda(T)$ is invertible, and in this case they are similar.

Throughout the remainder of this paper, we denote by

$$\delta(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) \,/ U^2|T| = |T|U^2\}.$$

It is well known that

$$T \text{ is quasinormal} \iff U|T| = |T|U.$$

Hence every quasinormal operator belongs to $\delta(\mathcal{H})$, but the converse does not hold.

We start by giving a condition under which an operator in $\delta(\mathcal{H})$ becomes quasinormal.

Proposition 1. *(see, e.g., [4, p. 291]) Let n be a positive integer and $T \in \delta(\mathcal{H})$, with polar decomposition $T = U|T|$. If $U^{2n+1} = I$, then T is quasinormal.*

In general

$$\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}, \tag{1}$$

does not hold. It would be certainly interesting to know which invertible operators in $\mathcal{B}(\mathcal{H})$ satisfy the equation(1).

Recently, the answer to this problem in the case of matrices was given by D. Pappas et al. as follows:

Theorem 1. *(see, e.g., [3, p. 4406]) Let $T \in \mathbb{C}^{n\times n}$ be invertible. It holds that*

$$\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1} \iff \lambda = \frac{1}{2} \text{ and } T = \Delta_\lambda(T).$$

We solve this problem in the case of $\lambda = 1$, for bounded linear operators.

Theorem 2. *(see, e.g., [4, p. 291]) Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. Then*

$$T \in \delta(\mathcal{H}) \iff \Delta_1(T^{-1}) = (\Delta_1(T))^{-1}.$$

Example 1. Theorem 2 is not valid when the Duggal transform is replaced by the Aluthge transform. To see this let $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H}\oplus\mathcal{H})$, where A and B are invertible positive operators such that $AB \neq BA$. Then T is invertible and

$$T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = U|T|$$

is the polar decomposition of T. Since $U^2 = I$, it follows that $U^2|T| = |T|U^2$ and so $T \in \delta(\mathcal{H}\oplus\mathcal{H})$. On the other hand, since

$$\Delta(T) = \begin{pmatrix} 0 & B^{\frac{1}{2}}A^{\frac{1}{2}} \\ A^{\frac{1}{2}}B^{\frac{1}{2}} & 0 \end{pmatrix}, \text{ we obtain } (\Delta(T))^{-1} = \begin{pmatrix} 0 & B^{-\frac{1}{2}}A^{-\frac{1}{2}} \\ A^{-\frac{1}{2}}B^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

So, we have

$$\Delta(T^{-1}) = |T^{-1}|^{\frac{1}{2}}U^*|T^{-1}|^{\frac{1}{2}} = |T^*|^{-\frac{1}{2}}U^*|T^*|^{-\frac{1}{2}} = \begin{pmatrix} 0 & A^{-\frac{1}{2}}B^{-\frac{1}{2}} \\ B^{-\frac{1}{2}}A^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Hence $\Delta(T^{-1}) \neq (\Delta(T))^{-1}$.

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