Embeddings properties of variable Besov-type spaces

Z. Zeghad
M'sila University, M'sila, Algeria
zgzoheyr@gmail.com

We say that $g\colon \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\rm loc}$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + \frac{1}{|x-y|})}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the log-*Hölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally*-log-*Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. We define the following class of variable exponents

$$\mathcal{P}^{\log} := \Big\{p \in \mathcal{P}\colon \frac{1}{p} \in C^{\log}\Big\},$$

which were introduced in [3, Section 2]. The mixed Lebesgue sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ was introduced by Almeida and Hästö in [1]. For $v \in \mathbb{Z}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$, let $Q_{v,m}$ be the dyadic cube in $\mathbb{R}^n$, $Q_{v,m} = \{(x_1, ..., x_n)\colon m_i \leq 2^v x_i < m_i + 1, i = 1, 2, ..., n\}$. For the collection of all such cubes we use

$$\mathcal{Q} := \{Q_{v,m}\colon v \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

For each cube Q, we denote its center by c_Q, its lower left-corner by $x_{Q_{v,m}} = 2^{-v}m$ of $Q = Q_{v,m}$ and its side length by $l(Q)$. For $r > 0$, we denote by rQ the cube concentric with Q having the side length $rl(Q)$. Furthermore, we put $v_Q = -\log_2 l(Q)$ and $v_Q^+ = \max(v_Q, 0)$. Select a pair of Schwartz functions Φ and φ such that

$$\mathrm{supp}\mathcal{F}\Phi \subset \overline{B(0,2)} \quad \text{and} \quad |\mathcal{F}\Phi(\xi)| \geq c \quad \text{if} \quad |\xi| \leq \frac{5}{3} \tag{1}$$

and

$$\mathrm{supp}\mathcal{F}\varphi \subset \overline{B(0,2)}\backslash B(0,1/2) \quad \text{and} \quad |\mathcal{F}\varphi(\xi)| \geq c \quad \text{if} \quad \frac{3}{5} \leq |\xi| \leq \frac{5}{3}, \tag{2}$$

where $c > 0$. We put $\varphi_v := 2^{vn}\varphi(2^v\cdot), v \in \mathbb{N}$.

Definition 1. Let $\alpha\colon \mathbb{R}^n \to \mathbb{R}$, $\tau\colon \mathbb{R}^n \to \mathbb{R}^+$ and $p, q \in \mathcal{P}_0$. Let Φ and φ satisfy (1) and (2), respectively. The Besov-type space $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{P\in\mathcal{Q}} \Big\| \Big(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\geq v_P^+}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty,$$

where φ_0 is replaced by Φ.

Remark 1. The definition of the spaces $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ is independent of the choices of Φ and φ.

For the spaces $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ introduced above we want to show some embedding theorems. We say a quasi-Banach space A_1 is continuously embedded in another quasi-Banach space A_2, $A_1 \hookrightarrow A_2$, if $A_1 \subset A_2$ and there is a $c > 0$ such that $\|f\|_{A_2} \leq c\,\|f\|_{A_1}$ for all $f \in A_1$. We begin with the following elementary embeddings.

Theorem 1. *Let $\alpha, \tau \in C^{\log}_{\rm loc}$, $\tau^- > 0$ and $p, q, q_0, q_1 \in \mathcal{P}^{\log}_0$.*
(i) If $q_0 \leq q_1$, then

$$\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

(ii) If $(\alpha_0 - \alpha_1)^- > 0$, then

$$\mathfrak{B}^{\alpha_0(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha_1(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

The proof can be obtained by using the same method as in [1, Theorem 6.1]. We next consider embeddings of Sobolev-type. It is well-known that

$$B^{\alpha_0,\tau}_{p_0,q} \hookrightarrow B^{\alpha_1,\tau}_{p_1,q},$$

if $\alpha_0 - \frac{n}{p_0} = \alpha_1 - \frac{n}{p_1}$, where $0 < p_0 < p_1 \leq \infty, 0 \leq \tau < \infty$ and $0 < q \leq \infty$ (see e.g. [2, Corollary 2.2]). In the following theorem we generalize these embeddings to variable exponent case.

Theorem 2. *Let $\alpha_0, \alpha_1, \tau \in C^{\log}_{\rm loc}$, $\tau^- > 0$ and $p_0, p_1, q \in \mathcal{P}^{\log}_0$ with $q^+ < \infty$. If $\alpha_0(\cdot) > \alpha_1(\cdot)$ and $\alpha_0(\cdot) - \frac{n}{p_0(\cdot)} = \alpha_1(\cdot) - \frac{n}{p_1(\cdot)}$ with $\left(\frac{p_0}{p_1}\right)^+ < 1$, then*

$$\mathfrak{B}^{\alpha_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha_1(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Remark 2. We would like to mention that

$$\mathfrak{B}^{\alpha_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha_0(\cdot)-\frac{n}{p_0(\cdot)},\tau(\cdot)}_{\infty,q(\cdot)}$$

if $\alpha_0, \tau \in C^{\log}_{\rm loc}, \tau^- > 0$ and $p_0, q, \tau \in \mathcal{P}^{\log}_0$, with $q^+ < \infty$.

Now we establish some further embedding of the spaces $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Theorem 3. *Let $\alpha, \tau \in C^{\log}_{\rm loc}, \tau^- > 0$ and $p, q \in \mathcal{P}^{\log}_0$ with $q^+ < \infty$. If $(p_2 - p_1)^+ \leq 0$, then*

$$\mathfrak{B}^{\alpha(\cdot)+n\tau(\cdot)+\frac{n}{p_2(\cdot)}-\frac{n}{p_1(\cdot)},0}_{p_2(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Remark 3. The proof of Theorem 1 − 3 are given in [4].

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