International Conference of Young Mathematicians
The Institute of Mathematics of the National Academy of Sciences of Ukraine
June 1–3, 2023, Kyiv, Ukraine

Darboux transformation of the indefinite Sturm–Liouville operator

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We study a self-adjoint Sturm-Liouville operator

$$\ell = -\frac{d^2}{dx^2} + q(x)$$

on the indefinite space $L_2((-a,a), sign(x))$, where a potential $q \in L_1^{loc}(-a,a)$ and $a \in \overline{\mathbb{R}}_+$.

Remark 1. $L_2((-a,a), sign(x))$ is a Krein space, where the indefinite inner product is defined by

$$[f,g] = \int_{-a}^{a} f(x)\overline{g(x)} sign(x) dx, \quad f,g \in L_2((-a,a), sign(x)).$$

Theorem 1. *Let $\ell = -\frac{d^2}{dx^2} + q(x)$ be the self-adjoint Sturm-Liouville operator on the indefinite space $L_2((-a,a), sign(x))$, where the potential q satisfies the following conditions:*

$$q \in L_1^{loc}(-a,a) \quad and \quad q(x) > 0, \quad a \in \overline{\mathbb{R}}_+. \tag{1}$$

Then the point spectrum of ℓ is

$$\sigma_p(\ell) = \{\lambda_n\}_{n=1}^{\infty} \cup \{\lambda_{-n}\}_{n=1}^{\infty}.$$

Moreover, the eigenvalues satisfy the following

$$\ldots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \ldots \quad .$$

The classical Darboux transformation of the Sturm–Liouville operators was studied in [1]. Now, we investigate Darboux transformation on $L_2((-a,a), sign(x))$. If we use coupling of the Sturm–Liouville operator on $L_2((-a,a), sign(x))$ and old results of the Darboux transformation, then we obtain the following theorem.

Theorem 2. *Let $\ell = -\frac{d^2}{dx^2} + q(x)$ be the self-adjoint Sturm–Liouville operator on $L_2((-a,a), sign(x))$, where the potential q satisfies* (1). *Let λ_1 and ϕ_1 be eigenvalue and eigenfunction of ℓ, respectively, (i.e. $\ell(\phi_1) = \lambda_1\phi_1(x)$) and $\lambda_{-2}, \lambda_{-1}, \lambda_1 \in \sigma_p(\ell)$, such that $|\lambda_{-1}| \leqslant \lambda_1 < |\lambda_{-2}|$. Then*

1. *ℓ admits the following factorization*

$$\ell = LR + \lambda_1,$$

where L and R are defined by

$$L = \frac{d}{dx} + \frac{\phi_1'(x)}{\phi_1(x)} \quad and \quad R = -\frac{d}{dx} + \frac{\phi_1'(x)}{\phi_1(x)}.$$

2. *The Darboux transformation of ℓ is the following self-adjoint Sturm–Liouville operator on $L_2((-a, a), sign(x))$*
$$\ell_1 = RL + \lambda_1.$$
Furthermore, the point spectrum of ℓ_1 can be found by
$$\sigma_p \ell_1 = \sigma_p(\ell) \setminus \{\lambda_{-1}, \lambda_1\}.$$

1. Matveev V. B., Salle M. A. Darboux transformations and Solitons. — Berlin: Springer, 1991, 120 p.