The approximation property for spaces of Lipschitz functions

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The theory of linear operator ideals between normed (or Banach) spaces have been developed by Albert Pietsch [4], and nowadays it is well established. A linear operator ideal $\mathcal{I}$ is a subclass of the class of all continuous linear operators, such that for every Banach spaces E and F, the set $\mathcal{I}(E,F)$ is a vector subspace of $\mathcal{L}(E,F)$ that is invariant by the composition of linear operators on the right or on the left and which contains the linear operators of finite rank.

Several classes of linears operator ideals, for example, the p-summing, p-nuclear, p-integral, compact and weakly compact operators have been developed and studied in the Lipschitz setting by several authors. In this talk we introduce the notion of Lipschitz operator ideal and we extend the notion and main results on the approximation property for Banach spaces to the case of metric spaces.

Recall that a pointed metric space X is a metric space with a base point in X denoted by 0. A map $T : X \longrightarrow E$ between metric space and Banach space is called Lipschitz if there is a positive constant C such that

$$\forall x, y \in X, \quad \|T(x) - T(y)\| \leq Cd(x,y).$$

For a Lipschitz map $T : X \longrightarrow E$ its Lipschitz constant is given by

$$Lip\,(T) = \sup\left\{\frac{\|T(x) - T(y)\|}{d(x,y)}, x \neq y\right\}.$$

We denote by

$$Lip_0(X,E) = \{T : X \longrightarrow Y \text{ Lipschitz}, \ \text{ with } T(0) = 0\}$$

a Banach space under the Lipschitz norm $Lip\,(.)$. For $E = \mathbb{K}$, we designate $Lip_0(X,\mathbb{K}) = Lip_0(X) = X^{\#}$. The space of all linear operators from E to F is denoted by $\mathcal{L}(E,F)$ and it is a Banach space with the usual supremum norm. It is clear that $\mathcal{L}(E,F)$ is a subspace of $Lip_0(E,F)$ and, in particular, E^* is a subspace of $E^{\#}$.One of the main tools that we will use is the Lipschitz-free Banach space of a metric space X, Æ(X) (also known as the Arens–Ells space). For $x \in X$, denote by δ_x the function $\delta_x : X^{\#} \longrightarrow \mathbb{K}$ defined as

$$\delta_x(f) = f(x), \ f \in X^{\#}.$$

Then Æ(X) is the closed linear span of $\{\delta_x, x \in X\}$ in $\left(X^{\#}\right)^*$. The dirac map $\delta_X \colon X \to$ Æ(X) is defined as $\delta_X(x)(f) = \delta_x(f)$ for all $x \in X$ and $f \in X^{\#}$. We summarize some basic properties concerning Lipschitz-free Banach spaces in the following lemma. This can be found for instance in [3]. Let $T : X \longrightarrow E$ be a Lipschitz map which preserves the base point; that is $T(0) = 0$. Then there is a unique bounded linear map $T_L :$ Æ$(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$. Furthermore

$$\|T_L\| = Lip(T).$$

Following [2], a mapping $T \in Lip_0(X,E)$ has Lipschitz finite dimensional rank if the linear hull of the set $\left\{\frac{T(x)-T(x')}{d(x,x')}, x, x' \in X, x \neq x'\right\}$ is a finite dimensional subspace of E. The set of all

Lipschitz finite rank mappings from pointed metric X space to Banach space E is denoted by $Lip_{0\mathcal{F}}(X, E)$.

The approximation property of Banach spaces was introduced by Alexander Grothendieck in the fifties. A Banach space E has the *approximation property*(AP for short) if the identity map $Id_E : E \to E$ can be approximated uniformly on compact sets by finite rank operators. Oja in 2012 introduced the natural version of the AP related to an arbitrary Banach operator ideal $\mathcal{I}$. A Banach space E has the $\mathcal{I}$-approximation property if, for every Banach space F, $\overline{\mathcal{F}(F, E)}^{\|\cdot\|_{\mathcal{I}}} = \mathcal{I}(F, E)$.

Now we are going to present the Lipschitz operator ideal and the approximation property of metric spaces.

Definition 1. [1] A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components

$$\mathcal{I}_{Lip}(X, E) := Lip_0(X, E) \cap \mathcal{I}_{Lip}$$

satisfy:

1. $\mathcal{I}_{Lip}(X, E)$ is a linear subspace of $Lip_0(X, E)$.
2. $Lip_{0\mathcal{F}}(X, E) \subset \mathcal{I}_{Lip}(X, E)$
3. The ideal property: if $S \in Lip_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $w \in \mathcal{L}(E, F)$, then the composition wTS is in $\mathcal{I}_{Lip}(Y, F)$.

Definition 2. [1] Given an operator ideal $\mathcal{I}$, a Lipschitz mapping $T \in Lip_0(X, E)$ belongs to the *composition Lipschitz operator ideal* $\mathcal{I} \circ Lip_0$, denoted $T \in \mathcal{I} \circ Lip_0(X, E)$, if there are a Banach space F, a Lipschitz operator $S \in Lip_0(X, F)$ and an operator $u \in \mathcal{I}(F, E)$ such that $T = u \circ S$.

Definition 3. [1] Let X and Y be pointed metric spaces, and $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ a composition Lipschitz operator ideal. A set $K \subseteq X$ is *relatively $\mathcal{I}$-Lipschitz compact* if there is a pointed metric space Y and a Lipschitz operator $S : Y \to Æ(X)$ in $\mathcal{I}_{Lip}$ such that $\delta_X(K) \subseteq S(M)$, where M is a compact subset of Y.

Our first aim now is to prove that the $\mathcal{I}$-Lipschitz approximation property is weaker than the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property when they can be compared

Theorem 1. *[1] Let $\mathcal{I}$ be an operator ideal. Let E be a Banach space with the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. Then it has the $\mathcal{I}$-Lipschitz approximation property as a metric space with respect to the set $\delta_E \circ Lip_{0\mathcal{F}}(E, E)$.*

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