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The estimates of the inner radii of symmetric non-overlapping domains

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Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be the Riemann sphere, and $r(B,a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to the point $a \in B$.

Consider the different non-overlapping domains $B_0, B_1, \ldots, B_n$ ($B_p \cap B_j = \emptyset$ for $p \neq j$, $p, j = \overline{0,n}$) such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \mathbb{C}$, $k = \overline{1,n}$, moreover domains $B_1, \ldots, B_n$ have symmetry with respect to unit circle, and for $\gamma \in (0, n]$ consider the value

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right), \tag{1}$$

Problem 1 (see [1])**.** For any fixed $\gamma \in (0, n]$ to find the maximum of the functional (1) and to show that this maximum is reached for some configuration of the domains B_k and points a_k, $k = \overline{0,n}$, which has n-fold symmetry.

This problem is one of the problems of the geometric function theory. The problem has a solution only if $\gamma \leqslant n$ as soon as $\gamma = n + \epsilon, \epsilon > 0$, the problem has no solution. Currently it still unsolved in general, only partial results are known (see, f.e. [2]).

The following theorem holds (prove see in [3]).

Theorem 1. *Let $n = \overline{4,7}$, $1 < \gamma \leqslant \gamma_n$, $\gamma_4 = 1,6$, $\gamma_5 = 1,65$, $\gamma_6 = 1,7$, $\gamma_7 = 1,77$. Then for any different system of points $\{a_k\}_{k=0}^n$ such that $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$ and for any different system of non-overlapping domains $\{B_k\}_{k=0}^n$ such that $a_0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \mathbb{C}$, $k = \overline{1,n}$, moreover domains $\{B_k\}_{k=1}^n$ have symmetry with respect to unit circle, the following inequality holds*

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{4}{n}\right)^{n} \frac{\left(\frac{2\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{2\gamma}{n^2}\right)^{\frac{n}{2} + \frac{\gamma}{n}}} \left(\frac{n - \sqrt{2\gamma}}{n + \sqrt{2\gamma}}\right)^{\sqrt{2\gamma}}.$$

Equality is attained if a_k and B_k, $k = \overline{0,n}$ are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + 2(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2}\, dw^2.$$

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