Some property of factorable Lipschitz quasi weakly and unconditionally p-nuclear operators

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Since the article of Farmer and Johnson [2], which deals with a class of Lipschitz operators, the p-summing Lipschitz operators, several authors have studied different classes of Lipschitz operators which, in some sense, extend linear Banach operator ideals. In this talk we introduce and investigate the factorable Lipschitz quasi unconditionally p-nuclear operators defined between a pointed metric space and a Banach space, showing that a Lipschitz map is factorable unconditionally quasi p-nuclear, if and only if its adjoint is unconditionally p-compact. The notion of factorable Lipschitz quasi weakly p-nuclear operators is introduced and showing that it coincides with the Lipschitz injective hull of the ideal of Lipschitz weakly p-nuclear operators.

Definition 1. [3] Let X be metric space and E be a Banach space. For $1 \leq p \leq \infty$ and $T \in Lip_0(X, E)$, we say that T is called strongly Lipschitz classical p-compact if T has a factorization $T = R \circ S$, where $S \in Lip_{0\mathcal{K}}(X, \ell_p)$ and $R \in \mathcal{K}(\ell_p, E)$ (also, T is called strongly Lipschitz (∞, p, p^*)-nuclear). For a pointed metric space X and a Banach space E the norm on the vector space $\mathcal{CK}^L_{st,p}(X, E)$ of all strongly Lipschitz classical p-compact operators from X to E is defined by

$$\|T\|_{\mathcal{CK}^L_{st,p}} := \inf \left\{ Lip(S)\|R\| : S \in Lip_{0\mathcal{K}}(X, \ell_p), R \in \mathcal{K}(\ell_p, E) \text{ and } T = R \circ S \right\}.$$

Definition 2. Let X be metric space and E be a Banach space. For $1 \leq p \leq \infty$ and $T \in Lip_0(X, E)$, we say that $T : X \to E$ is Lipschitz weakly p-nuclear operators if T can be written in the form

$$T(x) = \sum_{n=1}^{\infty} f_n(x) y_n, \forall x \in X, \tag{1}$$

where $(f_n)_n \in \ell_p^{L,\omega^*}(X^\#)(c_0(X^\#) \quad \text{when } p = \infty)$ and $(y_n)_n \in \ell^{\omega}_{p^*}(E)$. Also, the set of all Lipschitz weakly p-nuclear operators will be denoted by $\mathcal{N}^L_{\omega p}(X, E)$ and we set

$$\|T\|_{\mathcal{N}^L_{\omega p}} = \inf \|(f_n)_n\|_p^{L,\omega^*} \|(y_n)_n\|^{\omega}_{p^*},$$

with the infimum taken over all representations of T as in (1).

Definition 3. [3] A mapping $T \in Lip_0(X, E)$ is called factorable Lipschitz quasi unconditionally p-nuclear operator if there exists a sequence $(f_n)_n \in \ell_p^{L,\omega^*,u}(X^\#)$ such that

$$\left\| \sum_{j=1}^{m} \lambda_j (Tx_j - Tx'_j) \right\| \leq \left(\sum_{n=1}^{\infty} \left| \sum_{j=1}^{m} \lambda_j (f_n(x_j) - f_n(x'_j)) \right|^p \right)^{\frac{1}{p}}, \tag{2}$$

for all $x_j, x'_j \in X, \lambda_j \in \mathbb{K}$, $(1 \leq j \leq m)$.
We denote by $\mathcal{FN}^{LQ}_{u,p}(X, E)$ the space of all factorable quasi unconditionally$p$-nuclear Lipschitz mappings, we put

$$\|T\|_{\mathcal{FN}^{LQ}_{u,p}} = \inf \left\{ \|(f_n)_n\|_p^{L,\omega^*} : (f_n)_n \text{ satisfying } (2) \right\}.$$

Definition 4. A mapping $T \in Lip_0(X, E)$ is called factorable Lipschitz quasi weakly p-nuclear operator if there exists a sequence $(f_n)_n \in \ell_p^{L,\omega^*}(X^{\#})$ such that

$$\left\| \sum_{j=1}^{m} \lambda_j (Tx_j - Tx'_j) \right\| \leq \left(\sum_{n=1}^{\infty} \left| \sum_{j=1}^{m} \lambda_j (f_n(x_j) - f_n(x'_j)) \right|^p \right)^{\frac{1}{p}}, \tag{3}$$

for all $x_j, x'_j \in X, \lambda_j \in \mathbb{K}$, $(1 \leq j \leq m)$.
We denote by $\mathcal{FN}_{\omega p}^{LQ}(X, E)$ the space of all factorable quasi weakly p-nuclear Lipschitz mappings, we put

$$\|T\|_{\mathcal{FN}_{\omega,p}^{LQ}} = \inf \left\{ \|(f_n)_n\|_p^{L,\omega^*} : (f_n)_n \text{ satisfying (3)} \right\}.$$

In [1], the authors introduced the *injective hull* of a Banach Lipschitz operator ideal. For this, recall that for a Banach space E, we may consider the linear isometry $\iota_E \colon E \longrightarrow \ell_\infty(B_{E^*})$. Then, following [1, Definition 2.2 (1)], for a Banach Lipschitz operator ideal $\mathcal{I}_{Lip}$, a pointed metric space X and a Banach space E, a Lipschitz operator $T \colon X \to E$ belongs to $\mathcal{I}_{Lip}^{inj}(X, E)$ if and only if $\iota_E T$ belongs to $\mathcal{I}_{Lip}(X, \ell_\infty(B_{E^*}))$ with $\|T\|_{\mathcal{I}_{Lip}^{inj}} = \|\iota_E T\|_{\mathcal{I}_{Lip}}$. If $\mathcal{I}_{Lip}$ is a Banach Lipschitz operator ideal, then $\mathcal{I}_{Lip}^{inj}$ is also a Banach Lipschitz operator ideal.

Theorem 1. *Let X be pointed metric space, E be a Banach space , then $T \in (\mathcal{CK}_{st,p}^{L})^{inj}$ if and only if $T \in \mathcal{FN}_{u,p}^{LQ}(X, E)$.*

Theorem 2. *Let X be pointed metric space, E be a Banach space , then $T \in (\mathcal{N}_{\omega p}^{L})^{inj}$ if and only if $T \in \mathcal{FN}_{\omega,p}^{LQ}(X, E)$.*

Proposition 1. *For an operator $T \in Lip_0(X, E)$, $T^t : E^* \longrightarrow X^{\#}$ is unconditionally p-compact operator if and only if $T \in \mathcal{FN}_{u,p}^{LQ}(X, E)$ and $\|T\|_{\mathcal{FN}_{u,p}^{LQ}} = \|T^t\|_{\mathcal{K}_{u,p}}$. In other words, $\mathcal{FN}_{up}^{LQ}(X, E) = \mathcal{K}_{u,p}^{Lip-dual}(X, E)$ isometrically.*

Proposition 2. *For an operator $T \in Lip_0(X, E)$, $T^t : E^* \longrightarrow X^{\#}$ is weackly p-compact operator if and only if $T \in \mathcal{FN}_{\omega p}^{LQ}(X, E)$ and $\|T\|_{\mathcal{FN}_{\omega p}^{LQ}} = \|T^t\|_{\mathcal{W}_p}$. In other words, $\mathcal{FN}_{\omega p}^{LQ}(X, E) = \mathcal{W}_p^{Lip-dual}(X, E)$ isometrically.*

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