Crum transformation of the Laguerre operators

E. Svyatovets
Mykhailo Dragomanov Ukrainian State University, Kyiv, Ukraine
Misslizzi.s.v@gmail.com

We study a Crum transformation of the Laguerre operators. Recall some terms.

Definition 1 (see [4, Ch. 5])**.** The Laguerre polynomials $L_n(x,\alpha)$ are defined by

$$L_n(x,\alpha)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!},\quad n\in\mathbb{Z}_+ \text{ and } \alpha\in\mathbb{R}\backslash\mathbb{Z}_-.$$

If $\alpha>-1$, then the sequence of Laguerre polynomials $\{L_n(x,\alpha)\}_{n=0}^{\infty}$ is orthogonal in the Hilbert space $L_2(\mathbb{R}_+,w_\alpha)$, where the weight function w_α is defined by

$$w_\alpha(x)=x^\alpha e^{-x}$$

and $L_n(x,\alpha)$ are called the classical Laguerre polynomials.

If $\alpha<-1$ and $\alpha\notin\mathbb{Z}_-$, then the sequence of Laguerre polynomials $\{L_n(x,\alpha)\}_{n=0}^{\infty}$ is orthogonal in the Pontryagin space $\Pi(\alpha)$ (see [1]) and $L_n(x,\alpha)$ are called the nonclassical Laguerre polynomials.

In [1,2] the Laguerre operator was studied in the form $\ell=xy''+(\alpha+1-x)y'$. But, we study the self-adjoint Laguerre operator on $L_2(0,+\infty)$, which is defined by

$$\ell_\alpha=-\frac{d^2}{dx^2}+\frac{\alpha^2-\frac{1}{4}}{x^2}+x^2.$$

If $\alpha>-1$, then ℓ_α is called a classical Laguerre operator. If $\alpha<-1$ and $\alpha\notin\mathbb{Z}_-$, then ℓ_α is called a nonclassical Laguerre operator.

As is know [4, Ch. 5], the point spectrum of ℓ_α is

$$\sigma_p(\ell_\alpha)=\left\{\lambda_n|\lambda_n=2(1+\alpha)+4n,\quad n\in\mathbb{Z}_+\right\}$$

and eigenfunctions have the following representation

$$\phi_n(x,\alpha)=e^{-\frac{x^2}{2}}x^{\alpha+\frac{1}{2}}L_n(x^2,\alpha),\quad n\in\mathbb{Z}_+.$$

Definition 2 (see [3, Ch. 2])**.** Let $\ell=-\dfrac{d^2}{dx^2}+q$ be the self-adjoint Sturm-Liouville operator on $L_2(a,b)$ and let $y_1,...,y_n$ be the first nth eigenfunctions of ℓ. The Crum transformation of ℓ is called the following self-adjoint Sturm-Liouville operator

$$\tilde{\ell}=-\frac{d^2}{dx^2}+\tilde{q},$$

where the potential $\tilde{q}$ can be found by

$$\tilde{q}(x)=q(x)-2\frac{d}{dx}\left(\frac{W'(y_1,...,y_n)}{W(y_1,...,y_n)}\right),\quad W(y_1,...,y_n)=\begin{vmatrix} y_1(x) & \dots & y_n(x)\\ \vdots & \ddots & \vdots\\ y_1^{(n-1)}(x) & \dots & y_n^{(n-1)}(x)\end{vmatrix}.$$

Theorem 1. *Let ℓ_α be the self-adjoint Laguerre operator with $\alpha \notin \mathbb{Z}_-$. Then the Crum transformation of ℓ_α constructed by ϕ_0 and ϕ_1 is*

$$\tilde{\ell}_{\alpha,2} = -\frac{d^2}{dx^2} + \left(\frac{(\alpha+2)^2 - \frac{1}{4}}{x^2} + x^2 + 4\right).$$

Furthermore

$$\sigma_p(\tilde{\ell}_{\alpha,2}) = \sigma_p(\ell_\alpha)\backslash\{\lambda_0, \lambda_1\}$$

and the eigenfunctions of $\tilde{\ell}_{\alpha,2}$ are

$$\psi_n(x,\alpha) = \phi_n(x, \alpha+2).$$

Corollary 1. *Let ℓ_α be the nonclassical Laguerre operator with $\alpha \in (-3,-1)\backslash\{-2\}$. Then*

$$\tilde{\ell}_{\alpha,2} = \ell_{\alpha+2} + 4.$$

Theorem 2. *Let ℓ_α be the self-adjoint Laguerre operator with $\alpha \notin \mathbb{Z}_-$. Then the Crum transformation of ℓ_α constructed by ϕ_0, ϕ_1, ϕ_2 and ϕ_3 is*

$$\tilde{\ell}_{\alpha,4} = -\frac{d^2}{dx^2} + \left(\frac{(\alpha+4)^2 - \frac{1}{4}}{x^2} + x^2 + 8\right).$$

Furthermore

$$\sigma_p(\tilde{\ell}_{\alpha,4}) = \sigma_p(\ell_\alpha)\backslash\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$$

and the eigenfunctions of $\tilde{\ell}_{\alpha,4}$ are

$$\psi_n(x,\alpha) = \phi_n(x, \alpha+4).$$

Corollary 2. *Let ℓ_α be the nonclassical Laguerre operator with $\alpha \in (-5,-1)\backslash\{-4,-3,-2\}$. Then*

$$\tilde{\ell}_{\alpha,4} = \ell_{\alpha+4} + 8.$$

1. Derkach V. A. Extensions of Laguerre operators in indefinite inner product spaces. Math. Notes, 1998, **63**, No. 4, 449 – 459.
2. Kovalyov I. Darboux Transformation of the Laguerre Operator. Complex Anal. Oper. Theory, 2018, **12**, No. 3, 787 – 809.
3. Matveev V. B. and Salle M. A. Darboux transformations and Solitons. — Berlin: Springer, 1991, 120 p.
4. Szegö G. Orthogonal Polynomials. — Providence: American Mathematical Society (4th edn.), 1975, 432 p.