σ-MONOGENIC FUNCTIONS IN COMMUTATIVE ALGEBRAS

V. S. Shpakivskyi
Institute of Mathematics of the National Academy of Science of Ukraine, Kyiv, Ukraine
shpakivskyi86@gmail.com

Let $\mathbb{A}$ be an arbitrary n-dimensional $(1 \leqslant n < \infty)$ commutative associative algebra with unit over the field of complex number $\mathbb{C}$. E. Cartan proved that in $\mathbb{A}$ there exist a basis $\{I_k\}_{k=1}^n$ such that the first m basis vectors $I_1, I_2, \ldots, I_m$ are idempotents and another vectors $I_{m+1}, I_{m+2}, \ldots, I_n$ are nilpotents. The element $1 = I_1 + I_2 + \ldots + I_m$ is the unit of $\mathbb{A}$.

In the algebra $\mathbb{A}$ we consider the vectors $e_1, e_2, \ldots, e_d$, $2 \leqslant d \leqslant 2n$. Let these vectors have the following decomposition in the basis of the algebra:

$$e_j = \sum_{r=1}^{n} a_{jr}\, I_r\,, \quad a_{jr} \in \mathbb{C}, \quad j = 1, 2, \ldots, d. \tag{1}$$

Throughout this paper, we will assume that at least one of the vectors $e_1, e_2, \ldots, e_d$ is invertible. This condition ensures the uniqueness of the σ-derivative.

For the element $\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d$, where $x_1, x_2, \ldots, x_d \in \mathbb{R}$, the complex numbers

$$\xi_u := x_1a_{1u} + x_2a_{2u} + \cdots + x_da_{du}, \qquad u = 1, 2, \ldots, m$$

forms the spectrum of the point ζ.

Consider in the algebra $\mathbb{A}$ a linear span

$$E_d := \{\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d \colon x_1, x_2, \ldots, x_d \in \mathbb{R}\}$$

generated by the vectors $e_1, e_2, \ldots, e_d$ of $\mathbb{A}$.

Next, the assumption is essential: for each fixed $u \in \{1, 2, \ldots, m\}$ at least one of the numbers $a_{1u}, a_{2u}, \ldots, a_{du}$ belongs to $\mathbb{C} \setminus \mathbb{R}$.

We identify a domain Ω in the space $\mathbb{R}^d$ with the domain

$$\Omega := \{\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d \colon (x_1, x_2, \ldots, x_d) \in S\} \text{ in } E_d \subset \mathbb{A}.$$

Definition 1 ([1])**.** We will call the continuous function $\Phi\colon \Omega \to \mathbb{A}$ *monogenic* in the domain $\Omega \subset E_d$ if Φ is differentiable in the sense of Gâteaux at every point of this domain, that is, if for each $\zeta \in \Omega$ there exists an element $\Phi'(\zeta)$ of the algebra $\mathbb{A}$ such that the equality

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h\Phi'(\zeta) \quad \forall\, h \in E_d \tag{2}$$

holds. $\Phi'(\zeta)$ is called the *Gâteaux derivative* of the function Φ at the point ζ.

The theory of monogenic functions in commutative algebras is well developed in the works of the author and his colleagues S. A. Plaksa, S. V. Gryshchuk and R. P. Pukhtaievych. Monogenic functions are some analog of analytic functions in commutative algebras.

At the end of book [2], V. Kravchenko poses 5 open problems. In the fourth problem, Kravchenko points out the need to construct a pseudoanalytic function theory in multidimensional case.

Our work is an attempt to solve the Kravchenko problem in the case of any finite-dimensional commutative associative algebra. Namely, by developing the ideas of L. Bers and G. Polozhii, σ-monogenic functions will be introduce in any commutative associative algebra.

Let a function $\Phi\colon \Omega \to \mathbb{A}$ be of the form

$$\Phi(\zeta) = \sum_{k=1}^{n} U_k(x_1, x_2, \ldots, x_d)\, I_k\,. \tag{3}$$

Let σ be a collection of n $\mathbb{A}$-valued functions:

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n),$$

where $\sigma_k = \sigma_k(x_1, x_2, \ldots, x_d) = \sigma_k(\zeta)$, $k = 1, 2, \ldots, n$, is a function in $\mathbb{A}$.

For a vector $h = \sum\limits_{j=1}^{d} h_j e_j$, $h_j \in \mathbb{R}$, we denote

$$\Delta_{\varepsilon,h,\sigma}\Phi(\zeta) := \sum_{k=1}^{n} \sigma_k(\zeta)\Big(U_k(x_1 + \varepsilon h_1, \ldots, x_d + \varepsilon h_d) - U_k(x_1, \ldots, x_d)\Big)\, I_k\,.$$

Definition 2. We will call the continuous function $\Phi\colon \Omega \to \mathbb{A}$ *σ-monogenic* in the domain $\Omega \subset E_d$ if for each $\zeta \in \Omega$ there exists an element $\Phi'_\sigma(\zeta)$ of the algebra $\mathbb{A}$ such that for every $h \in E_d$ the equality

$$\lim_{\varepsilon \to 0+0} \frac{\Delta_{\varepsilon,h,\sigma}\Phi(\zeta)}{\varepsilon} = h\, \Phi'_\sigma(\zeta) \tag{4}$$

holds. $\Phi'_\sigma(\zeta)$ is called *σ-derivative* of the function Φ at the point ζ.

Remark 1. If for all $k = 1, 2, \ldots, n$ $\sigma_k \equiv 1$, then definition 2 coincides with definition 1, i. e. 1-monogenic function is monogenic.

Remark 2. If $\mathbb{A} \equiv \mathbb{C}$ and for special choice of σ_1, σ_2 the definition (4) coincides with the definition of pseudoanalytic function in the sense of Bers.

Necessary and sufficient conditions for σ-monogeneity have been established. In some low-dimensional algebras, with a special choice of σ, the representation of σ-monogenic functions is obtained using holomorphic functions of a complex variable. We proposed the application of σ-monogenic functions with values in two-dimensional biharmonic algebra to representation of solutions of two-dimensional biharmonic equation. The announced results are published in the paper [3].

1. Mel'nichenko I. P. The representation of harmonic mappings by monogenic functions. Ukrain. Math. J., 1975, **27**, No. 5, 499 – 505.
2. Kravchenko V. V. Applied pseudoanalytic function theory. Birkhäuser, Series: Frontiers in Mathematics, Basel, 2008.
3. Shpakivskyi V. S. σ-monogenic functions in commutative algebras. Proceedings of the International Geometry Center, 2023, **16**, No. 1, 17 – 41.