# Oscillations in the discrete case

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Let $(a_n)_{n\geqslant 0}$ be an arbitrary (finite or infinite) sequence of real numbers. We consider functions defined as

$$f(x) = \sum_{n\geqslant 0} a_n \cdot \chi_{[n,n+1)}(x),$$

where $\chi_A$ is the characteristic function of the set $A$. For the function $f$ we denote by $f_{[p,q]}$ and by $\Omega(f,[p,q])$ the arithmetic mean on the interval $[p,q]$ and, respectively, the mean oscillation on the interval $[p,q]$,

$$f_{[p,q]} = \frac{1}{q-p}\sum_{i=p}^{q-1} a_i, \qquad \Omega(f,[p,q]) = \frac{1}{q-p}\sum_{i=p}^{q-1}\left|a_i - f_{[p,q]}\right|.$$

The mean oscillations were studied firstly in paper [1] and now they have numerous applications in many sections of mathematics.

Our main results can be formulated as the following theorems.

**Theorem 1** ([2]). *Let the function* $f(x) = \sum_{n=0}^{Q-1} a_n \chi_{[n,n+1)}(x)$ *be monotone. Then*

$$\max_{\{p,q\in\mathbb{Z}\colon\, 0\leqslant p<q\leqslant Q\}} \Omega\left(f,[p,q]\right) = \max_{\{r\in\mathbb{Z}\colon\, 0\leqslant r\leqslant Q\}} \max\left\{\Omega\left(f,[0,r]\right), \Omega\left(f,[r,Q]\right)\right\}.$$

The result of Theorem 1 is new, as far as we know. Analogous result is well-known in the continuous case, when the supremum is taken over all intervals. Its classical proof is based on the intermediate value property. We find the new method of proof, which allows one to carry out proofs for set systems without the intermediate value property.

**Theorem 2.** *Let the function* $f(x) = \sum_{n=0}^{\infty} a_n \chi_{[n,n+1)}(x)$ *be monotone. Then*

$$\max_{\{p,q\in\mathbb{Z}\colon\, 0\leqslant p<q<\infty\}} \Omega\left(f,[p,q]\right) = \max_{\{r\in\mathbb{Z}\colon\, 0<r\}} \Omega\left(f,[0,r]\right).$$

The result of Theorem 2 is also new, as far as we know. We have found a new proof method that allows us to obtain this result, which is also well-known in the continuous case.

The results of Theorems 1 and 2 can be useful in the study of the $BMO$ space and equimeasurable rearrangements, in particular, to obtain discrete analogues of the John–Nirenberg theorem.

1. John F., Nirenberg L. On functions of bounded mean oscillation. Comm. Pure Appl. Math., 1961, **14**, No. 3, 415 – 426.
2. Korenovskyi A. O., Shanin R. V. On One Property of the Mean-Arithmetic Oscillations of a Monotone Sequence. Ukrain. Math. J., 2022, **74**, No. 4, 586 – 596.