Sampling recovery in the uniform norm

K. V. Pozharska

Institute of Mathematics of the National Academy of Science of Ukraine, Kyiv, Ukraine

Chemnitz University of Technology, Chemnitz, Germany

pozharska.k@gmail.com

We study the recovery of functions in the uniform norm based on function evaluations. We obtain worst case error bounds for general classes of functions, also in L_p-norm, in terms of the best L_2-approximation from a given nested sequence of subspaces combined with bounds on the the Christoffel function of these subspaces.

Let D be an arbitrary set, μ be a measure on D. By $L_p = L_p(\mu)$, $1 \leqslant p < \infty$, we denote the space of complex-valued functions that are p-integrable with respect to μ, and by $L_\infty = L_\infty(\mu)$ the space of essentially bounded functions on D. Moreover, we denote by $B(D)$ the space of bounded, complex-valued functions on D with the sup-norm.

For a class $F \subset L_p$ and $n \in \mathbb{N}$, we define the n-th linear sampling number in L_p by

$$g_n^{\mathrm{lin}}(F, L_p) \ := \inf_{x_1,\dots,x_n \in D,\ \varphi_1,\dots,\varphi_n \in L_p} \ \sup_{f \in F} \left\| f - \sum_{i=1}^{n} f(x_i)\, \varphi_i \right\|_p .$$

This is the minimal worst case error that can be achieved with linear algorithms based on at most n function values, if the error is measured in L_p.

Our goal is to identify properties of classes F that allow for the existence of good linear sampling recovery algorithms for the uniform norm. The main assumption here is that there exists a sequence of subspaces V_n that are "good" for L_2-approximation. In addition, the (inverse of the) Christoffel function, which is sometimes also called spectral function, defined by

$$\Lambda_n \ := \ \Lambda(V_n) \ := \sup_{f \in V_n,\, f \neq 0} \|f\|_\infty / \|f\|_2, \quad n \in \mathbb{N},$$

will play an important role. Note that for any orthonormal basis $\{b_k\}_{k=1}^n$ of the n-dimensional space V_n we have $\Lambda_n^2 = \big\| \sum_{k=1}^n |b_k|^2 \big\|_\infty$.

One of the main results can be stated as follows.

Theorem 1. *Let μ be a finite measure on a set D, $(V_n)_{n=1}^\infty$ be a nested sequence of subspaces of $B(D)$ of dimension n, and F be a separable subset of $B(D)$. Assume that*

$$\Lambda(V_n) \ \lesssim \ n^\beta \qquad \text{and} \qquad \sup_{f \in F} \inf_{g \in V_n} \|f - g\|_2 \lesssim n^{-\alpha} (\log n)^\gamma$$

for some $\alpha > \beta \geqslant 1/2$ and $\gamma \in \mathbb{R}$. Then, for all $1 \leqslant p \leqslant \infty$,

$$g_n^{\mathrm{lin}}(F, L_p) \ \lesssim \ n^{-\alpha + (1-2/p)_+ \beta} (\log n)^\gamma$$

with $a_+ := \max\{a, 0\}$.

Note, that the condition on β is no restriction because for a finite measure, where L_2-norm is dominated by L_∞-norm, we always have $\beta \geqslant 1/2$. See also [1, Thm. 12] for the general statement with more explicit constants. In the talk we will discuss some examples that show that this bound is often sharp up to constants.

There are numerous results in the literature on uniform approximation, often for specific classes F and explicit algorithms. The advantage of Theorem 1 is its generality, as it comes without any possibly redundant assumption. However, this result is non-constructive. The underlying (least squares) algorithm is based on a random construction and subsampling based on the solution of the Kadison-Singer problem.

To measure the quality of our upper bounds, one may take the Gelfand widths of the class F as a benchmark. The n-th Gelfand width of F in L_p is defined by

$$c_n(F, L_p) := \inf_{\psi\colon \mathbb{C}^n \to L_p,\ N\colon F \to \mathbb{C}^n \ \text{linear}} \sup_{f \in F} \big\| f - \psi \circ N(f) \big\|_p$$

and measures the worst case error of the optimal (possibly non-linear) algorithm using n pieces of arbitrary linear information.

Theorem 2. *There are absolute constants $b, c \in \mathbb{N}$ such that the following holds. Let μ be a measure on a set D and $H \subset L_2$ be a reproducing kernel Hilbert space with kernel*

$$K(x,y) = \sum_{k=1}^{\infty} \sigma_k^2 \, b_k(x) \, \overline{b_k(y)}, \qquad x, y \in D,$$

where $(\sigma_k)_{k=1}^{\infty} \in \ell_2$ is a non-increasing sequence and $\{b_k\}_{k=1}^{\infty}$ is an orthonormal system in L_2 such that there is a constant $B > 0$ with $\Lambda_n^2 \leqslant Bn$ for all $n \in \mathbb{N}$. Under these conditions we have $H \subset L_\infty$ and

$$g_{bn}^{\text{lin}}(H, L_\infty)^2 \leqslant cB \sum_{k>n} \sigma_k^2.$$

In particular, if μ is a finite measure, then

$$g_{bn}^{\text{lin}}(H, L_\infty) \leqslant \sqrt{cB \cdot \mu(D)} \cdot c_n(H, L_\infty).$$

This result has been observed independently in [2, Thm. 2.2] for the trigonometric system. It represents a special case of a more general result for reproducing kernel Hilbert spaces where we do not need the uniform boundedness of the basis, see [1, Thm. 8].

Clearly, Theorem 2 can also be applied for other orthonormal systems than the trigonometric monomials, including the Haar or the Walsh system, certain wavelets, the Chebychev polynomials, and the spherical harmonics, if μ is their corresponding orthogonality measure.

In all these cases, we obtain that linear sampling algorithms are (up to constants) as powerful as arbitrary non-linear algorithms using general linear measurements. Note that we do not require any decay of the Gelfand width c_n.

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