Mappings contracting perimeters of triangles

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The Contraction Mapping Principle was established by S. Banach in his dissertation (1920) and published in 1922 [1]. Although the idea of successive approximations in a number of concrete situations (solution of differential and integral equations, approximation theory) had appeared earlier in the works of P. L. Chebyshev, E. Picard, R. Caccioppoli, and others. S. Banach was the first who formulated this result in a correct abstract form suitable for a wide range of applications. After a century, the interest of mathematicians around the world to fixed-point theorems is still very high. This is confirmed by the appearance in recent decades of numerous articles and monographs devoted to the fixed point theory and its applications.

In this work we consider a new type of mappings in metric spaces which can be characterized as mappings contracting perimeters of triangles. It is shown that such mappings are continuous. The fixed point theorem for such mappings is proved and the classical Banach fixed-point theorem is obtained like a simple corollary. An example of a mapping contractive perimeters of triangles which is not a contraction mapping is constructed.

Definition 1. Let (X, d) be a metric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *mapping contracting perimeters of triangles* on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \leqslant \alpha(d(x, y) + d(y, z) + d(x, z)) \tag{1}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Note that the requirement for $x, y, z \in X$ to be pairwise distinct is essential. One can see that otherwise this definition is equivalent to the definition of contraction mapping.

Proposition 1. *Mappings contracting perimeters of triangles are continuous.*

Theorem 1. *Let (X, d), $|X| \geqslant 3$, be a complete metric space and let the mapping $T\colon X \to X$ satisfy the following two conditions:*

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is a mapping contracting perimeters of triangles on X.

Then T has a fixed point. The number of fixed points is at most two.

Remark 1. Suppose that under the supposition of the theorem the mapping T has a fixed point x^* which is a limit of some iteration sequence $x_0, x_1 = Tx_0, x_2 = Tx_1, \ldots$ such that $x_n \neq x^*$ for all $n = 1, 2, \ldots$. Then x^* is a unique fixed point.

Example 1. Let us construct an example of the mapping T contracting perimeters of triangles which has exactly two fixed points. Let $X = \{x, y, z\}$, $d(x, y) = d(y, z) = d(x, z) = 1$, and let $T\colon X \to X$ be such that $Tx = x$, $Ty = y$ and $Tz = x$. One can easily see that conditions (i) and (ii) of Theorem 1 are fulfilled.

Example 2. Let us show that condition (i) of Theorem 1 is necessary. Let $X = \{x, y, z\}$, $d(x, y) = d(y, z) = d(x, z) = 1$, and let $T\colon X \to X$ be such that $Tx = y$, $Ty = x$ and $Tz = x$. One can easily see that condition (ii) of Theorem 1 is fulfilled but T does not have any fixed point.

Let (X, d) be a metric space. Then a mapping $T\colon X \to X$ is called a *contraction mapping* on X if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \tag{2}$$

for all $x, y \in X$.

Corollary 1. ***(Banach fixed-point theorem)*** *Let (X, d) be a nonempty complete metric space with a contraction mapping $T\colon X \to X$. Then T admits a unique fixed point.*

Proof. For $|X| = 1, 2$ the proof is trivial. Let $|X| \geqslant 3$. Suppose that there exists $x \in X$ such that $T(Tx) = x$. Consequently, $d(x, Tx) = d(Tx, x) = d(Tx, T(Tx))$, which contradicts to (2). Thus, condition (i) of Theorem 1 holds. Let $x, y, z \in X$ be pairwise distinct. By (2) we obtain $d(T(x), T(y)) \leqslant \alpha d(x, y)$, $d(T(y), T(z)) \leqslant \alpha d(y, z)$ and $d(T(x), T(z)) \leqslant \alpha d(x, z)$ which immediately implies condition (ii) of Theorem 1. This completes the proof of existence of fixed point.

The uniqueness can be shown in a standard way. □

Example 3. Let us construct an example of a mapping $T\colon X \to X$ contracting perimeters of triangles that is not a contraction mapping for a metric space X with $|X| = \aleph_0$. Let $X = \{x^*, x_0, x_1, \ldots\}$ and let a be positive real number. Define a metric d on X as follows:

$$d(x, y) = \begin{cases} a/2^{\lfloor i/2 \rfloor}, & \text{if } x = x_i, y = x_{i+1}, i = 0, 1, 2, ..., \\ d(x_i, x_{i+1}) + \cdots + d(x_{j-1}, x_j), & \text{if } x = x_i,\ y = x_j,\ i + 1 < j, \\ 4a - d(x_0, x_i), & \text{if } x = x_i,\ y = x^*, \\ 0, & \text{if } x = y, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Define a mapping $T\colon X \to X$ as $Tx_i = x_{i+1}$ for all $i = 0, 1, \ldots$ and $Tx^* = x^*$. Since $d(x_{2n}, x_{2n+1}) = d(Tx_{2n}, Tx_{2n+1})$, $n = 0, 1, 2...$, using (2) we see that T is not a contraction mapping. It is possible to show that inequality (1) holds for every three pairwise distinct points from the space X.

1. Banach S. Sur les opérations dans les ensembles abstraits et leur application auxéquations intégrales. Fund. Math., 1922, **3**, 133 – 181.