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A note on Schauder bases of mixed norm spaces with an application to fractal Schauder bases

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Schauder bases are widely used in the study of partial differential equations, which arise in many areas of science and engineering. Schauder bases provide a powerful tool for understanding the properties of solutions to these equations and constructing numerical methods for their solution. On the other hand, mixed norm function spaces, a generalization of the classical Lebesgue-type spaces, have its root in the seminal article by Benedek and Panzone [Duke Math. J., 28 (1961), pp. 301-324]. In contrast to the classical case, the integrability exponent in the mixed norm Lebesgue spaces may differ for different variables. This versatility makes it a better tool for partial differential equations and pseudo-differential operators with varying physical properties for each variable. In this talk, we shall establish the relationship between the Schauder bases of function spaces with mixed norm spaces and their classical counterparts. As an application, we shall demonstrate *fractal Schauder bases* in function spaces with the mixed norm.

Let $\overrightarrow{P} = (p_1, p_2, \ldots, p_n) \in (0,\infty]^n$ and (E_i, S_i, μ_i), $1 \le i \le n$ be σ-finite measure spaces. Let (E, S, μ) be the associated product measure space and $f : E \to \mathbb{R}$ be a measurable function. Define the quantity $\|f\|_{\overrightarrow{P}}$ as follows

$$\|f\|_{\overrightarrow{P}} = \left(\int \ldots \left(\int \left(\int \left|f(x_1,\ldots,x_n)\right|^{p_1} \mathrm{d}\mu_1\right)^{\frac{p_2}{p_1}} \mathrm{d}\mu_2\right)^{\frac{p_3}{p_2}} \ldots \mathrm{d}\mu_n\right)^{\frac{1}{p_n}},$$

with the usual modification if $p_k = \infty$ for any $k = 1, 2, \ldots . n$.

Definition 1. For $\overrightarrow{P} \in (0,\infty]^n$, the mixed norm Lebesgue space $\mathcal{L}^{\overrightarrow{P}}(E)$ is defined as follows

$$\mathcal{L}^{\overrightarrow{P}}(E) = \Big\{ f : E \to \mathbb{R}; \quad f \quad \text{is a measurable and } \|f\|_{\overrightarrow{P}} < \infty \Big\}.$$

Now, let $f \in \mathcal{L}^{\overrightarrow{P}}(E)$ and $l = (l_1, \ldots, l_n)$ be a multi-index the weak derivative of f is denoted by $D^l(f)$. We use the notation $|l| = \sum_{k=1}^m l_k$

Definition 2. Let m be a non-negative integer and $\overrightarrow{P} \in (0,\infty]^n$, the mixed norm Sobolev space $\mathcal{W}^{m,\overrightarrow{P}}(E)$ is defined as follows

$$\mathcal{W}^{m,\overrightarrow{P}}(E) = \Big\{ f \in \mathcal{L}^{\overrightarrow{P}}(E); \quad D^l(f) \in \mathcal{L}^{\overrightarrow{P}}(E),\ \forall\ |l| \le m \Big\}.$$

We equip the mixed Sobolev space with the norm $\|f\|_{m,\overrightarrow{P}} := \sum_{|l|\le m} \|D^l(f)\|_{\overrightarrow{P}}$. It is well-known that (see, for instance, [1]) the spaces $\left(\mathcal{L}^{\overrightarrow{P}}(E), \|\cdot\|_{\overrightarrow{P}}\right)$ and $\left(\mathcal{W}^{m,\overrightarrow{P}}(E), \|\cdot\|_{m,\overrightarrow{P}}\right)$ are quasi-Banach spaces for $\overrightarrow{P} \in (0,\infty]^n$ and indeed a Banach space if $\overrightarrow{P} \in [1,\infty]^n$. Further, if we take $\overrightarrow{P} = (p, p, \ldots, p) \in (0,\infty]^n$, then $\left(\mathcal{L}^{\overrightarrow{P}}(E), \|.\|_{\overrightarrow{P}}\right)$ and $\left(\mathcal{W}^{m,\overrightarrow{P}}(E), \|\cdot\|_{m,\overrightarrow{P}}\right)$ coincide with their classical counter parts $\left(\mathcal{L}^p(E), \|\cdot\|_p\right)$ and $\left(\mathcal{W}^{m,p}(E), \|\cdot\|_{m,p}\right)$, respectively. The curious readers may refer to [?, ?] and the recent survey article [3] for explicating these spaces.

Let $\overrightarrow{P} \in [1, \infty)^n$. Set $p_{\max} = \max\{p_k : 1 \leq k \leq n\}$ and $p_{\min} = \min\{p_k : 1 \leq k \leq n\}$. for all measurable functions $f : E \to \mathbb{R}$.

Now, let $\mathcal{B} = \{\psi_n\}_{n=1}^{\infty} \subset \mathcal{L}^{p_{\max}}(E) \cap \mathcal{L}^{p_{\min}}(E)$ be such that it is Schauder basis for both $\mathcal{L}^{p_{\max}}(E)$ and $\mathcal{L}^{p_{\min}}(E)$. In the following theorem we shall prove that $\mathcal{B}$ is also a Schauder basis for the mixed Lebesgue space $\mathcal{L}^{\overrightarrow{P}}(E)$.

Theorem 1. *Let $\mathcal{B} = \{\psi_n\}_{n=1}^{\infty} \subset \mathcal{L}^{p_{\max}}(E) \cap \mathcal{L}^{p_{\min}}(E)$ be such that it is Schauder basis for both $\mathcal{L}^{p_{\max}}(E)$ and $\mathcal{L}^{p_{\min}}(E)$. Then $\mathcal{B}$ is also a Schauder basis for the mixed Lebesgue space $\mathcal{L}^{\overrightarrow{P}}(E)$.*

We prove a similar result for the mixed Sobolev spaces and establish the existence of *fractal Schauder bases* for these spaces as an interlude.

1. Benedek A., Panzone R., The space L_p with mixed norm, Duke Math. J., 1961, 28, 301–324.
2. Burenkov V.I., Viktorova N.B., The embedding theorem for Sobolev spaces with mixed norm for limit exponents, Math. Notes, 1996, 59, 45–51.
3. Huang L., Yang D., On function spaces with mixed norms — a survey, J. Math. Study, 2021, 54, 3, 262–336.