On Coefficient Bounds and Fekete-Szegő inequality for a Certain Family of Holomorphic and Bi-Univalent Functions Defined by (M,N)-Lucas Polynomials

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In this presentation we use the (M,N)-Lucas Polynomials to introduce a new family of holomorphic and bi-univalent functions which involve a linear combination between Bazilevič functions and β-pseudo-starlike function defined in the unit disk $\mathbb{D}$. We also establish upper bounds for the second and third coefficients of functions that belong to this new family and we discuss the Fekete-Szegő problem.

The Lucas Polynomials plays an important role in a diversity of disciplines as the mathematical, statistical, physical and engineering sciences.

We define the family $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$ as follows

Definition 1. For $0\leq\lambda\leq 1$; $\alpha\geq 0$; $\beta\geq 1$ let $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$ denote the subclass of Σ such that

$$(1-\lambda)\frac{z^{1-\alpha}f'(z)}{(f(z))^{1-\alpha}}+\lambda\frac{z\left(f'(z)\right)^{\beta}}{f(z)}\prec T_L\left(M,N;x,z\right)-1$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(f^{-1}(w)\right)'}{\left(f^{-1}(w)\right)^{1-\alpha}}+\lambda\frac{w\left(\left(f^{-1}(w)\right)'\right)^{\beta}}{f^{-1}(w)}\prec T_L\left(M,N;x,w\right)-1.$$

In particular, if we choose $\alpha=\lambda=0$ or $\lambda=\beta=1$ in Definition 1, we have $\mathscr{L}_{MN}(0,0,\beta;x)=\mathscr{L}_{MN}(1,\alpha,1;x):=P_\sigma(0;x)$ for the family of functions $f\in\Sigma$

$$\frac{zf'(z)}{f(z)}\prec T_L\left(M,N;x,z\right)-1$$

and

$$\frac{w\left(f^{-1}(w)\right)'}{f^{-1}(w)}\prec T_L\left(M,N;x,w\right)-1.$$

If $M(x)=1, N(x)=0$ then $\dfrac{zf'(z)}{f(z)}\prec T_L\left(1,0;x,z\right)-1=\dfrac{1}{1-z}$.

If $M(x)=2x, N(x)=-1$ then $\dfrac{zf'(z)}{f(z)}\prec T_L\left(2x,-1;x,z\right)-1=\dfrac{1}{1-2xz+z^2}$.

Theorem 1. *For $0\leq\lambda\leq 1$, $\alpha\geq 0$ and $\beta\geq 1$, let f belongs to the family $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$ and $N(x)\neq 0$.*

Let denote

$$\Omega(\lambda,\alpha,\beta)=(1-\lambda)(\alpha+1)+\lambda(2\beta-1),$$

$$E(\lambda,\alpha,\beta,M(x),N(x))=$$

$$= \frac{\sqrt{2}\,|M(x)|\,\sqrt{|M(x)|}}{\sqrt{\left|\left[(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)-2\Omega^2(\lambda,\alpha,\beta)\right]M^2(x)-4\Omega^2(\lambda,\alpha,\beta)N(x)\right|}}$$

and

$$F(\lambda,\alpha,\beta,M(x)) = \frac{|M(x)|}{\Omega(\lambda,\alpha,\beta)}\,.$$

Then

$$|a_2| \le min\left\{E(\lambda,\alpha,\beta,M(x),N(x)),F(\lambda,\alpha,\beta,M(x))\right\}$$

and

$$|a_3| \le \frac{M^2(x)}{\Omega^2(\lambda,\alpha,\beta)} + \frac{|M(x)|}{(1-\lambda)(\alpha+2)+\lambda(3\beta-1)}.$$

We can also obtain

$$|a_2| \le \frac{|L_{M,N,1}(x)|}{(1-\lambda)(\alpha+1)+\lambda(2\beta-1)} \le \frac{|M(x)|}{(1-\lambda)(\alpha+1)+\lambda(2\beta-1)}.$$

In the next theorem, we discuss "the Fekete-Szegő Problem" for the family $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$.

Theorem 2. *For $0 \le \lambda \le 1$, $\alpha \ge 0$, $\beta \ge 1$ and $\delta \in \mathbb{R}$, let $f \in \mathcal{A}$ belongs to the family $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$. Then*

$$\left|a_3-\delta a_2^2\right| \le \begin{cases} \frac{|M(x)|}{(1-\lambda)(\alpha+2)+\lambda(3\beta-1)}; & for\ \ |\delta-1| \le \frac{1}{2[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)]}\times \\ & \times\left|(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)-2\Omega^2(\lambda,\alpha,\beta)-\frac{4\Omega^2(\lambda,\alpha,\beta)N(x)}{M^2(x)}\right|, \\ \frac{2|M(x)|^3|\delta-1|}{|[(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)-2\Omega^2(\lambda,\alpha,\beta)]M^2(x)-4\Omega^2(\lambda,\alpha,\beta)N(x)|}; & \\ & for\ \ |\delta-1| \ge \frac{1}{2[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)]}\times \\ & \times\left|(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)-2\Omega^2(\lambda,\alpha,\beta)-\frac{4\Omega^2(\lambda,\alpha,\beta)N(x)}{M^2(x)}\right| \end{cases}$$

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