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Generalized c-almost periodic functions defined on vertical strips in the complex plane

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As a generalization of purely periodic functions defined on the set of the real numbers, the Danish mathematician H. Bohr introduced the concept of almost periodicity during the 1920's of the last century.

Definition 1. A continuous function $f : \mathbb{R} \to X$, where X is a complex Banach space whose norm is denoted by $||\cdot||$, is said to be almost periodic (in Bohr's sense) if for every $\varepsilon > 0$ there corresponds a number $l = l(\varepsilon) > 0$ such that every open interval of length l contains a number τ satisfying

$$||f(x+\tau) - f(x)|| \leq \varepsilon, \ \forall x \in \mathbb{R}.$$

The number τ described above is called an ε-translation number of f and, equivalently, the property of almost periodicity means that the set of all ε-translation numbers of f is relatively dense (r.d.) on the real line. See also [2-4], [9] for more information on these spaces of functions. Also, in the course of time, this theory has experimented an increasing interest in order to extend and generalize the various types of Bohr's concept. In particular, the first generalizations were given by W. Stepanov $(1889 - 1950)$, who succeeded in removing the continuity restrictions and characterize this new class in terms of mean values over integrals of fixed length (see [1]).

Definition 2. Given $1 \leq p < \infty$, a locally integrable map $f : \mathbb{R} \to X$ is Stepanov-p-almost periodic (S^p-almost periodic) if and only if for every $\varepsilon > 0$ there corresponds a relatively dense set of real numbers $\{\tau\}$ satisfying

$$\sup_{x\in\mathbb{R}} \left(\int_x^{x+1} ||f(t+\tau) - f(t)||^p dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

A generalization of these functions was given by H. Weyl $(1885 - 1955)$.

Definition 3. Given $1 \leq p < \infty$, a locally integrable map $f : \mathbb{R} \to X$ is Weyl-p-almost periodic (W^p-almost periodic) if and only if for every $\varepsilon > 0$ there corresponds a relatively dense set of real numbers $\{\tau\}$ satisfying

$$\overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} ||f(t+\tau) - f(t)||^p dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

On the other hand, in connection with the notions of almost periodicity, Khalladi et al. [5] have recently introduced and analyzed the classes of c-almost periodic functions (which is denoted by $AP_c(\mathbb{R}, X)$) where c is a non-zero complex number.

Definition 4. Let $c \in \mathbf{C}\setminus\{0\}$. A continuous function $f : \mathbb{R} \to X$ is said to be c-almost periodic if for every $\varepsilon > 0$ there corresponds a relatively dense set of real numbers $\{\tau\}$ satisfying

$$||f(x+\tau) - cf(x)|| \leq \varepsilon, \ \forall x \in \mathbb{R}.$$

Also, by following the same ideas as above, these authors in [6] have studied and analyzed the classes of S_c^p-almost periodicity and W_c^p-almost periodicity.

Furthermore, the concept of almost periodicity defined on a vertical strip of the complex plane of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$, was theorized in [2], and it has been widely studied in the literature as an extension of the real case.

In this work, we develop and extend all of these concepts (c-almost periodicity, S_c^p-almost periodicity and W_c^p-almost periodicity) to the case of functions defined on vertical strips of the complex plane. Some of the the main properties of these new classes of functions are summarized as follows (see [7-8] for more information).

Proposition 1. *Let $1 \leq p < \infty$, $c \in \mathbb{C}\backslash\{0\}$ and $f : U \to \mathbb{C}$ a c-almost periodic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Then $AP_c(U, \mathbb{C}) = AP_{1/c}(U, \mathbb{C})$, $S_c^p(U, \mathbb{C}) = S_{1/c}^p(U, \mathbb{C})$ and $W_c^p(U, \mathbb{C}) = W_{1/c}^p(U, \mathbb{C})$.*

Proposition 2. *Let $1 \leq p < \infty$, $c \in \mathbb{C}\backslash\{0\}$ and $m \in \mathbb{Z}\backslash\{0\}$. Consider a vertical strip of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, (with $-\infty \leq \alpha < \beta \leq \infty$). Then $AP_c(U, \mathbb{C}) \subset AP_{c^m}(U, \mathbb{C})$, $S_c^p(U, \mathbb{C}) \subset S_{c^m}^p(U, \mathbb{C})$ and $W_c^p(U, \mathbb{C}) \subset W_{c^m}^p(U, \mathbb{C})$.*

Proposition 3. *Let $1 \leq p < \infty$and $c \in \mathbb{C}\backslash\{0\}$. Consider a vertical strip of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$ (with $-\infty \leq \alpha < \beta \leq \infty$). Then*

i) If $\frac{\arg c}{2\pi} \in \mathbb{Q}$, then $AP_c(U, \mathbb{C}) \subset AP_{|c|^q}(U, \mathbb{C})$, $S_c^p(U, \mathbb{C}) \subset S_{|c|^q}^p(U, \mathbb{C})$ and $W_c^p(U, \mathbb{C}) \subset W_{|c|^q}^p(U, \mathbb{C})$, where $q \in \mathbb{N}$ verifies $\frac{\arg c}{2\pi} = \frac{r}{q}$ for a certain $r \in \mathbb{Z}$ such that $\gcd(r, q) = 1$.

ii) If $\frac{\arg c}{\pi} \notin \mathbb{Q}$ and $|c| = 1$, then $AP_c(U, \mathbb{C}) \subset AP(U, \mathbb{C})$, $S_c^p(U, \mathbb{C}) \subset S^p(U, \mathbb{C})$. Moreover, if $f \in W_c^p(U, \mathbb{C})$ is W^p-bounded in every vertical substrip of U, then f is also in $W^p(U, \mathbb{C})$.

Theorem 1. *Given $c \in \mathbb{C}\backslash\{0\}$, let $f : U \to \mathbb{C}$ be a c-almost periodic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Then the family of vertical translates $\{f_h(z) : h \in \mathbb{R}\}$ (with $f_h(z) := f(z + ih)$, $z \in U$) is relatively compact on any vertical strip of U.*

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