Coupled Fixed Point Results for nonlinear contraction in b-Fuzzy Metric Space Endowed with Graph with Application

K. Mebarki[1], **A. Boudaoui**[2]

[1]Laboratory of Mathematics Modeling and Applications, University of Adrar, Algeria

[2]Laboratory of Mathematics Modeling and Applications, University of Adrar, Algeria

kha.mebarki@univ-adrar.edu.dz,ahmedboudaoui@univ-adrar.edu.dz

In 1987, Guo and Lakshmikantham [1] introduced the notion of coupled fixed point. By using this notion, Bhaskar and Lakshmikantham [2] gave sufficient conditions to solve some differential equations by introducing and proving many nice results for coupled fixed points.

In this presentation, we endowed a complete b-fuzzy metric space with a graph. We derive some new coupled fixed point theorems under some conditions. Then we apply our results to give sufficient conditions to guarantee an existence of a continuous solution for a system of fractional differential equation.

Let $(X, M, *, G)$ stands to a complete b-fuzzy metric space with constant $s \geq 1$ (introducing by Sedghi and Shobe [3]) such that $a * a \geq a^2$ and $\lim_{t\to\infty} M(x, y, t) = 1$, endowed with directed graph G such that $V(G) = X$, $E(G) \supseteq \Delta$ and G has no parallel edges. Further, we endow the product space $X \times X$ by another graph denoted also by G, such that

$$((x, y), (u, v)) \in E(G) \Leftrightarrow (x, u) \in E(G) \text{ and } (v, y) \in E(G),$$

for any $(x, y), (u, v) \in X \times X$.

We denote by Ω the set of function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ that meets all of the following criteria

1. φ is nondecreasing;
2. for all $a \in \mathbb{R}^+$ and $t \in \mathbb{R}^+$ we have $\varphi(at) = a\varphi(t)$;
3. $\sum_{i=0}^{\infty} \varphi^i(t)$ converges for all $t > 0$.

Definition 1. The mapping $T : X \times X \to X$ is called φ-fuzzy contraction if there exist $\varphi \in \Omega$ such that

1. for all $x, y, u, v \in X$, T is edge preserving, i.e.,
$$((x, y), (u, v)) \in E(G) \text{ then } ((T(x, y), T(y, x)), (T(u, v), T(v, u))) \in E(G);$$
2. for all $(x, y), (u, v) \in X \times X$ such that $((x, y), (u, v)) \in E(G)$,
$$M(T(x, y), T(u, v), \varphi(t)) \geq M(x, u, st)^{\frac{1}{2}} * M(y, v, st)^{\frac{1}{2}}.$$

We proved the following results.

Theorem 1. *On $(X, M, *)$, suppose that T is continuous mapping and φ-fuzzy contraction mapping. If there exist $x_0, y_0 \in X$ such that $((x_0, y_0), (T(x_0, y_0), T(y_0, x_0))) \in E(G)$, then T possesses a coupled fixed point.*

The continuity of T in Theorem 1 can be discarded by adding some new conditions. Assume that (X, d, G) possess the following property

1. For any $\{x_n\}_{n\in\mathbb{N}}$ in X such that $(x_n, x_{n+1}) \in E(G)$ and $\lim\limits_{n\to+\infty} x_n = x$, then $(x_n, x) \in E(G)$.
2. For any $\{x_n\}_{n\in\mathbb{N}}$ in X such that $(x_{n+1}, x_n) \in E(G)$ and $\lim\limits_{n\to+\infty} x_n = x$ then $(x, x_n) \in E(G)$.

Theorem 2. *Endowed (X, M, G) with the previous property. Suppose that T is φ-fuzzy contraction. If there exist x_0, $y_0 \in X$ such that $((x_0, y_0), (T(x_0, y_0), T(y_0, x_0))) \in E(G)$, then T possesses a coupled fixed point.*

Theorem 3. *In addition to the hypothesis of both Theorem 1 and Theorem 2, suppose that for every (x, y), $(x^*, y^*) \in X \times X$ there exists $(u, v) \in X \times X$ such that $((x, y), (u, v)) \in E(G)$ and $((x^*, y^*), (u, v)) \in E(G)$. Then T has unique coupled fixed point.*

Then, we studied the existence of a solution to the following system.

$$D^\alpha x(t) = f(t, x(t), y(t)), \quad D^\alpha y(t) = f(t, y(t), x(t)), \quad t \in J, \tag{1}$$

$$x(0) = x_0 = y(0). \tag{2}$$

Here the symbol D^α denotes the Caputo fractional derivative of order $\alpha \in (0, 1)$, $J := [0, L]$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying the following assumptions

1. f is contionous;
2. For all $x, y, u, v \in \mathbb{R}$, with $x \leq u$ and $v \leq y$ we have
$$f(t, x, y) \leq f(t, u, v), \quad \text{for all } t \in J;$$
3. For each $t \in J$, $x, y, u, v \in \mathbb{R}$, $x \leq u$ and $v \leq y$, we have
$$|f(t, x, y) - f(t, u, v)|^2 \leq \frac{1}{8}\left(|x - u|^2 + |y - v|^2\right);$$
4. We suppose that $K = \frac{L^{2\alpha-1}}{\Gamma(\alpha)^2} < 1$.

Theorem 4. *Consider the system (1)-(2) and suppose that the previous assumptions are satisfied. Assume that there exists $(u_0, v_0) \in X \times X$ such that*

$$u_0(t) \leq x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, u_0(s), v_0(s)) ds$$

and

$$v_0(t) \geq x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, v_0(s), u_0(s)) ds, \ t \in J.$$

Then, there exists a unique solution of the system (1)-(2).

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